EXERCISE 4 IN INTRODUCTION TO REPRESENTATION THEORY

DMITRY GOUREVITCH

Fourier transform for finite groups.

Let C be a finite commutative group, $n = |C|$. We will denote by $\widehat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\widehat{C})$ by $\mathcal{F}(u)(\psi) = \sum u(g)\psi(g)$.

(1) (P) Show that if we define an L^2-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n||u||^2$.

(2) (P) Using the Plancherel formula prove the following Theorem(Gauss). Fix a non-trivial multiplicative character χ and a nontrivial additive character ψ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$. Then $|\Gamma| = q^{1/2}$.

Induction

(3) (a) $H = \{e\}$, $\mathrm{Ind}_H^G(\mathbb{C}) = F(G)$.
   (b) For any $H, \mathrm{Ind}_H^G(\mathbb{C}) = F(G/H)$.
   (c) For any character χ of H, $\mathrm{Ind}_H^G(\chi) = \{f \in F(G) : f(gh) = \chi(h^{-1})f(g)$.

(4) (a) For $H < G$ and $\pi_1, \pi_2 \in Rep(H)$,

$$\mathrm{Ind}_H^G(\pi_1 \oplus \pi_2) = \mathrm{Ind}_H^G(\pi_1) \oplus \mathrm{Ind}_H^G(\pi_2).$$

   (b) For $H_1 < H_2 < G$ and $\pi \in Rep(H)$,

$$\mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} \pi = \mathrm{Ind}_{H_1}^G \pi$$

(5) (P) Let G be a finite group, D its subgroup and χ a character of D. Consider the induced representation $\pi = Ind_D^G(\chi)$. Show that π is irreducible iff the following condition holds:
(*) For any $g \in G$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to D and $\chi(x) \neq \chi(y)$.

(6) (P) Let G be a finite group, Z its central subgroup and χ a character of Z. Denote by $Irr(G)_\chi$ the set of equivalence classes of irreducible representations of G on which Z acts via the character with the central character χ.
   (a) Compute $\sum_{\sigma \in Irr(G)_\chi} \dim^2 \sigma$.

Date: February 26, 2015.

(b) Explain how to find the size of the set $Irr(G)_\chi$. In particular show that this size is maximal when χ is a trivial character.

Equivariant sheaves

(7) (P)
 (a) Let X be a free G-set. Show that the category $Sh_G(X)$ is canonically equivalent to the category $Sh(G\backslash X)$. If you have done this correctly you should be able to deduce from this the following more general statement:
 (b) Let R be a group which contains G as a normal subgroup. Consider the quotient group $Q = R/G$. Suppose we have an R-set X that is free as a G-set. Show that the category $Sh_R(X)$ is canonically equivalent to the category $Sh_Q(G\backslash X)$.

(8) Let $\nu : X \to Y$ and let $\mathcal{F}_1, \mathcal{F}_2 \in Sh(X)$, $\mathcal{G}_1, \mathcal{G}_2 \in Sh(Y)$, $\phi : \mathcal{F}_1 \to \mathcal{F}_2$, $\psi : \mathcal{G}_1 \to \mathcal{G}_2$. Define natural maps $\nu_*(\phi) : \nu_*(\mathcal{F}_1) \to \nu_*(\mathcal{F}_2)$ and $\nu^*(\psi) : \nu^*(\mathcal{G}_1) \to \nu^*(\mathcal{G}_2)$.

(9) (P) The following structures on $\mathcal{F}$ are equivalent:
 (a) An equivariant structure
 (b) For any $x \in X$ and $g \in G$ - a linear map $\pi(g)_x : \mathcal{F}_x \to \mathcal{F}_{gx}$ such that for $g_1, g_2 \in G$, $\pi(g_1 g_2)_x = \pi(g_1) \circ \pi(g_2)_x$.
 (c) An isomorphism of sheaves $\alpha : a^*(\mathcal{F}) \approx \mathfrak{p}_2^*(\mathcal{F})$, where $p_2, a : G \times X \to X$ are the projection to the second coordinate and the action respectively, that satisfies the following condition:
 (*) Consider the set $Z = G \times G \times X$ and two morphisms $q, b : Z \to X$, defined by $q(g, g', x) = x$ and b(g, g', x) $= gg'x$. The morphism α induces two morphisms of sheaves $\beta, \gamma : q^*(\mathcal{F}) \to b^*(\mathcal{F})$. The condition on α is that these two morphisms are equal.

 Such diagramatic or categorical definition of an equivariant sheaf is very useful since it works well if we want to generalize the notion of an equivariant sheaf to other situations of similar flavor.

(10) ($\square$) There is a possibility to describe the categories of sheaves in more algebraic way. Let X be a finite set. Consider the algebra $F(X)$ of F-valued functions on X with usual (pointwise) multiplication.
 (a) Show that the category $Sh(X)$ is canonically equivalent to the category of $F(X)$-modules.
 (b) There exists a similar, but a little more sophisticated, description of the category Sh(X) for infinite sets X. Try to give such description.
 (c) Using (10a) give an algebraic description of the category $Sh_G(X)$.

(11) (P) Let G be a finite group and $N \lhd G$ a normal subgroup. The group G acts on N via conjugation and hence it acts on the set $I = Irr(N)$.

Definition 1. *A G-equivariant sheaf $\mathcal{F}$ on the set I is called special if for any point $\sigma \in I$ the action of the group N on the fiber $\mathcal{F}_\sigma$ is isotypical of type σ.*

Show that the subcategory of $Sh_G(I)$ consisting of special sheaves is naturally equivalent to the category $Rep(G)$.

(12) Let (π, G, V) be an irreducible representation of G. Suppose we know that the restriction $\pi|_N$ is not an isotypical representation. Show that in this case π is induced, i.e. there exists a subgroup $H < G$ containing N and an irreducible representation ρ of H such that π is isomorphic to $Ind_H^G(\rho)$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo.html