EXERCISE 4 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) (P) Find the characters of all the irreducible representations of the symmetric group Sym_4.
(2) (P) A representation is called cyclic if it is generated by some vector. Show that a representation is cyclic if and only if it is multiplicity free.

Classification of irreducible representations of S_n. Note that conjugate classes in S_n = partitions of n, i.e. sets $(\alpha_1, ..., \alpha_k)$ of natural numbers s.t. $\alpha_1 + ... + \alpha_k = n$ and $\alpha_1 \geq ... \geq \alpha_k$.

Let X be a set of size n and $G = \mathrm{Sym}(X) = S_n$. Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, ..., \alpha_k)$. Denote by X_α the set of all decompositions of the set X to subsets $X_1, .., X_k$ s.t. $|X_i| = \alpha_i$.

Definition 1. $T_\alpha := F(X_\alpha), \quad T'_\alpha := sgn \cdot T_\alpha.$

Definition 2. *Denote by α^* the partition given by $\alpha^*_i := |\{j : \alpha_j \leq i\}$.*

Let us introduce the lexicographical ordering on the set of partitions.

(3) Show that α^* is a partition and $(\alpha^*)^* = \alpha$.
(4) (*) Show that

$$\langle T_\alpha, T'_\beta \rangle = \begin{cases} 0, & \alpha > \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

This implies that T_α and T'_α have a unique joint irreducible component U_α and that these components are different for different α. This gives a classification of all irreducible representations of S_n. We will give here a formula for $\dim U_\alpha$, that we will prove later using Gelfand pairs:

$$\dim U_\alpha = \frac{n! \prod_{i<j}(l_i - l_j)}{l_1!...l_k!},$$

where $l_i = \alpha_i + k - i$, $i = 1, ..., k$.

Fourier transform for finite commutative groups. Let C be a finite commutative group, $n = |C|$. We will denote by $\widehat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\widehat{C})$ by $\mathcal{F}(u)(\psi) = \sum u(g)\psi(g)$.

(5) (P) Show that if we define an L^2-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n||u||^2$.

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(6) (P) Using the Plancherel formula prove the following Theorem(Gauss). Fix a non-trivial multiplicative character χ and a nontrivial additive character ψ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$. Then $|\Gamma| = q^{1/2}$.

Induction

(7) (a) $H = \{e\}$, $\mathrm{Ind}_H^G(\mathbb{C}) = F(G)$.
(b) For any $H, \mathrm{Ind}_H^G(\mathbb{C}) = F(G/H)$.
(c) For any character χ of H, $\mathrm{Ind}_H^G(\chi) = \{f \in F(G) : f(gh) = \chi(h^{-1})f(g)$.

(8) (a) For $H < G$ and $\pi_1, \pi_2 \in Rep(H)$,
$$\mathrm{Ind}_H^G(\pi_1 \oplus \pi_2) = \mathrm{Ind}_H^G(\pi_1) \oplus \mathrm{Ind}_H^G(\pi_2).$$
(b) For $H_1 < H_2 < G$ and $\pi \in Rep(H)$,
$$\mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} \pi = \mathrm{Ind}_{H_1}^G \pi$$

(9) (P) Let G be a finite group, D its subgroup and χ a character of D. Consider the induced representation $\pi = Ind_D^G(\chi)$. Show that π is irreducible if and only if the following condition holds:
(*) For any $g \in G$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to D and $\chi(x) \neq \chi(y)$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo2.html