Fourier transform for finite groups.
Let $C$ be a finite commutative group, $n = |C|$. We will denote by $\hat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\hat{C})$ by $\mathcal{F}(u)(\psi) = \sum u(g)\psi(g)$.

(1) (P) Show that if we define an $L^2$-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n ||u||^2$.

(2) (P) Using the Plancherel formula prove the following Theorem (Gauss). Fix a non-trivial multiplicative character $\chi$ and a nontrivial additive character $\psi$ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$. Then $|\Gamma| = q^{1/2}$.

Induction

(1) (P) Let $G$ be a finite group, $D$ its subgroup and $\chi$ a character of $D$. Consider the induced representation $\pi = Ind_G^D(\chi)$. Show that $\pi$ is irreducible iff the following condition holds:
(*) For any $g \in G$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to $D$ and $\chi(x) \neq \chi(y)$.

(2) (P) Let $G$ be a finite group, $Z$ its central subgroup and $\chi$ a character of $Z$. Denote by $Irr(G)_\chi$ the set of equivalence classes of irreducible representations of $G$ on which $Z$ acts via the character with the central character $\chi$.
(a) Compute $\sum_{\sigma \in Irr(G)_\chi} \dim \sigma$.
(b) Explain how to find the size of the set $Irr(G)_\chi$. In particular show that this size is maximal when $\chi$ is a trivial character.

$\textbf{c-solvable groups}$

$\textbf{Definition 1.}$ A representation induced from a character of a subgroup is called monomial.

$\textbf{Definition 2.}$ Let us call a group $G$ $\text{c-solvable}$ (which means cyclicly solvable) if there exists a sequence of normal subgroups $N_0 < N_1 < ... < N_k = G$ starting with the trivial subgroup $N_0$ such that each quotient group $N_i/N_{i-1}$ is cyclic.
(3) Show that any subgroup and quotient group of a c-solvable group is c-solvable. Show that any finite nilpotent group is c-solvable.

(4) (P) Let $G$ be a c-solvable finite group. Then any irreducible representation $\pi$ of $G$ is monomial.
   Hint: Can assume that the group is not commutative and the representation $\pi$ is faithful, i.e. no group element acts trivially. Let $Z < G$ denote the center. Choose a normal cyclic subgroup $C < G/Z$ and lift it to a normal commutative subgroup $N < G$. Show that $\pi|_N$ is not isotypic, and use Mackey theory to prove by induction on the (minimal) length of the chain.

(5) (P) Suppose we know that a group $G$ has a commutative normal subgroup $N$ such that the group $G/N$ is c-solvable. Show that any irreducible representation $\sigma$ of $G$ is monomial.