EXERCISE 5 IN INTRODUCTION TO REPRESENTATION THEORY

DMITRY GOUREVITCH

Fourier transform for finite groups.

Let C be a finite commutative group, $n = |C|$. We will denote by $\widehat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\widehat{C})$ by $\mathcal{F}(u)(\psi) = \sum u(g)\psi(g)$.

(1) (P) Show that if we define an L^2-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n||u||^2$.

(2) (P) Using the Plancherel formula prove the following Theorem(Gauss). Fix a non-trivial multiplicative character χ and a nontrivial additive character ψ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$. Then $|\Gamma| = q^{1/2}$.

Induction

(1) (P) Let G be a finite group, D its subgroup and χ a character of D. Consider the induced representation $\pi = Ind_D^G(\chi)$. Show that π is irreducible iff the following condition holds:
(*) For any $g \in G$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to D and $\chi(x) \neq \chi(y)$.

(2) (P) Let G be a finite group, Z its central subgroup and χ a character of Z. Denote by $Irr(G)_\chi$ the set of equivalence classes of irreducible representations of G on which Z acts via the character with the central character χ.
   (a) Compute $\sum_{\sigma \in Irr(G)_\chi} \dim^2 \sigma$.
   (b) Explain how to find the size of the set $Irr(G)_\chi$. In particular show that this size is maximal when χ is a trivial character.

c-solvable groups

Definition 1. *A representation induced from a character of a subgroup is called monomial.*

Definition 2. *Let us call a group G c-solvable(which means cyclicly solvable) if there exists a sequence of normal subgroups $N_0 < N_1 < \ldots < N_k = G$ starting with the trivial subgroup N_0 such that each quotient group N_i/N_{i-1} is cyclic.*

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(3) Show that any subgroup and quotient group of a c-solvable group is c-solvable. Show that any finite nilpotent group is c-solvable.

(4) (P) Let G be a c-solvable finite group. Then any irreducible representation π of G is monomial.
Hint: Can assume that the group is not commutative and the representation π is faithful, i.e. no group element acts trivially. Let $Z < G$ denote the center. Choose a normal cyclic subgroup $C < G/Z$ and lift it to a normal commutative subgroup $N < G$. Show that $\pi|_N$ is not isotypic, and use Mackey theory to prove by induction on the (minimal) length of the chain.

(5) (P) Suppose we know that a group G has a commutative normal subgroup N such that the group G/N is c-solvable. Show that any irreducible representation σ of G is monomial.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/RepTheo3.html