EXERCISE 5 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Barak has got an advanced game, where a usual game cube was replaced by a dodecahedron with numbers $1, ..., 12$ on its faces. It is not known how these numbers are distributed on the faces. Each time he lost, he replaced the number on each face by the average of its neighbors. Can you say, approximately but independent on the original distribution of the numbers, what numbers will be written on the faces after 30 losses? What is the precision of your answer?

Equivariant sheaves

(2) (P)
   (a) Let X be a free G-set. Show that the category $Sh_G(X)$ is canonically equivalent to the category $Sh(G\backslash X)$. If you have done this correctly you should be able to deduce from this the following more general statement:
   (b) Let R be a group which contains G as a normal subgroup. Consider the quotient group $Q = R/G$. Suppose we have an R-set X that is free as a G-set. Show that the category $Sh_R(X)$ is canonically equivalent to the category $Sh_Q(G\backslash X)$.

(3) Let $\nu : X \to Y$ and let $\mathcal{F}_1, \mathcal{F}_2 \in Sh(X)$, $\mathcal{G}_1, \mathcal{G}_2 \in Sh(Y)$, $\phi : \mathcal{F}_1 \to \mathcal{F}_2$, $\psi : \mathcal{G}_1 \to \mathcal{G}_2$. Define natural maps $\nu_*(\phi) : \nu_*(\mathcal{F}_1) \to \nu_*(\mathcal{F}_2)$ and $\nu^*(\psi) : \nu^*(\mathcal{G}_1) \to \nu^*(\mathcal{G}_2)$.

(4) (P) The following structures on $\mathcal{F}$ are equivalent:
   (a) An equivariant structure
   (b) For any $x \in X$ and $g \in G$ - a linear map $\pi(g)_x : \mathcal{F}_x \to \mathcal{F}_{gx}$ such that for $g_1, g_2 \in G$, $\pi(g_1 g_2)_x = \pi(g_1) \circ \pi(g_2)_x$.
   (c) An isomorphism of sheaves $\alpha : a^*(\mathcal{F}) \approx \mathfrak{p}_2^*(\mathcal{F})$, where $p_2, a : G \times X \to X$ are the projection to the second coordinate and the action respectively, that satisfies the following condition:
   (*) Consider the set $Z = G \times G \times X$ and two morphisms $q, b : Z \to X$, defined by $q(g, g', x) = x$ and b(g, g', x) $= gg'x$. The morphism α induces two morphisms of sheaves $\beta, \gamma : q^*(\mathcal{F}) \to b^*(\mathcal{F})$. The condition on α is that these two morphisms are equal.

Such diagramatic or categorical definition of an equivariant sheaf is very useful since it works well if we want to generalize the notion of an equivariant sheaf to other situations of similar flavor.

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(5) (□) There is a possibility to describe the categories of sheaves in more algebraic way. Let X be a finite set. Consider the algebra $F(X)$ of F-valued functions on X with usual (pointwise) multiplication.
  (a) Show that the category $Sh(X)$ is canonically equivalent to the category of $F(X)$-modules.
  (b) There exists a similar, but a little more sophisticated, description of the category Sh(X) for infinite sets X. Try to give such description.
  (c) Using (5a) give an algebraic description of the category $Sh_G(X)$.

(6) (P) Let G be a finite group and $N \triangleleft G$ a normal subgroup. The group G acts on N via conjugation and hence it acts on the set $I = Irr(N)$.

Definition 1. *A G-equivariant sheaf $\mathcal{F}$ on the set I is called special if for any point $\sigma \in I$ the action of the group N on the fiber $\mathcal{F}_\sigma$ is isotypical of type σ.*

Show that the subcategory of $Sh_G(I)$ consisting of special sheaves is naturally equivalent to the category $Rep(G)$.

(7) (*) Let (π, G, V) be an irreducible representation of G. Suppose we know that the restriction $\pi|_N$ is not an isotypical representation. Show that in this case π is induced, i.e. there exists a subgroup $H < G$ containing N and an irreducible representation ρ of H such that π is isomorphic to $Ind_H^G(\rho)$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo2.html