Theory of Automorphic forms.

Joseph Bernstein **Spring 2018**

Course description.

This is a first part of a year long course on the theory of Automorphic forms and Automorphic Representations. This theory is now one of the focal points of Mathematics. It has very many applications in different areas of Number Theory, Physics, Combinatorics and so on.

This theory is in the process of constant development and it is not easy to formulate what are the goals of the theory. I will try to describe basic structures of the theory and main ideas developed in it. I will also try to illustrate them by examples of applications.

I will try to describe the development of this theory in quasi-historical manner, starting with the classical theory of modular forms.

In Spring semester 2018 I will mostly discuss the classical theory of modular functions. I will also describe how to pass to the contemporary approach to automorphic forms based on automorphic representations of adelic groups.

I hope to continue this course in Fall of 2018.

I am planning to outline main conjectural structures of the theory evolving around Langlands' program, Generalized Riemann Hypothesis and Ramanujan conjecture.

One should realize that though in many cases we can formulate rather precise conjectures describing the behavior of L-functions and automorphic representations what we can prove about these conjectures is just scratching the surface.

Books. In my exposition I will use many sources. Here are some of them.

1. Book "1-2-3 of modular forms".
2. Book "Introduction to Langlands program"

Prerequisites. Basic notions of Functions of Complex variables, of Algebraic Geometry, of Representation Theory and of Number Theory.

Every time introducing new tools for the study of automorphic forms I will try to describe them in some detail and list main results. However it is clear that previous familiarity with these tools would be quite useful.

Home assignments. I will be giving problem assignments weekly. These problem assignments are the integral part of the course - they will contain many important points for which there is not enough time in the course itself.

The grades for home assignments will be a factor in the final grade for the course.

Syllabus.

1. Classical theory of modular forms for the group $SL(2, \mathbf{Z})$.

Relation with elliptic curves, Eisenstein series, Structure of the algebra of modular forms.

Fourier expansion of modular forms.

Different constructions of modular forms.

Discriminant function and cusp forms. Properties of coefficients $\tau(n)$.

Digression on Dirichlet L-functions.

Hecke operators and Hecke L-functions.

Peterson scalar product.

2. Maass forms and related L-functions.

3. Modular functions with respect to congruence subgroups. New forms and their L-functions. Weil's inverse theorem.

4. Modular forms with respect to co-compact subgroups. Statement of the Jacquet-Langlands correspondence.

6. Introduction to Hilbert modular forms.

7. Introduction to Siegel modular forms.

8. Automorphic forms and Representation Theory.

Digression on representations of the group $SL(2, \mathbf{R})$. Interpretation of automorphic forms in terms of automorphic representations.

9. Adelic automorphic representations.

Adeles.

Tate's thesis and L-functions

Passing to adelic automorphic representations.

Hecke L-function of an automorphic representation.

10. Langlands' L-function. Class field theory and Langlands program.

11. Langlands' dual group. Langlands program and functoriality conjecture.

12. Relation of Langlands program to Artin and Ramanujan conjectures.