
Let $K$ be a $p$-adic field. Fix its algebraic closure $\bar{K}$ and denote by $C$ its completion in norm topology.

Denote by $K_\infty \subset K$ the field extension of $K$ by all roots of 1 of degrees $p^n$, and denote by $L$ the completion of $K_\infty$ in $C$.

We denote $G = \text{Gal}(\bar{K}/K)$ and consider the closed subgroup $H = \text{Gal}(\bar{K}/K_\infty) \subset G$. We denote by $\Gamma$ the quotient group $\Gamma = \text{Gal}(K_\infty/K) = G/H$.

9.1. Generalities on descent. Consider, more generally, the situation where a topological group $G$ acts on a topological field $C$. Let $H$ be a closed normal subgroup of $G$, $\Gamma = G/H$, $K = C^G$, $L = C^H$.

Let $\text{Rep}(G, C)$ denote the category of finite dimensional $C$-vector spaces with continuous semi-linear action of the group $G$. Similarly consider the category $\text{Rep}(\Gamma, L)$.

We say that there is a descent from $C$ to $L$ if these categories are naturally equivalent.
In general case, consider the pair of adjoint functors
\[ R : \text{Rep}(G, C) \rightarrow \text{Rep} \Gamma, L \] and
\[ I : \text{rep}([\text{gam}, L] \rightarrow \text{Rep}(G, C) \]
given by
\[ R(V) = V^H, \quad I(W) = C \otimes_L W. \]

We have canonical morphisms of functors \( i : \text{Id} \rightarrow R \cdot I \)
and \( j : I \cdot R \rightarrow \text{Id} \).

Functor \( I \) clearly preserves the dimension. We will see
that always \( \dim(R(V)) \leq \dim(V) \).

**Proposition 9.1.1.** The following conditions are equivalent

(i) Functors \( I \) and \( R \) are mutually inverse equivalences of categories

(ii) Functor \( R \) preserves dimensions.

Indeed, functor \( R \) is left exact. If it preserves the dimension then it is exact and conservative.

Consider the adjunction morphism \( j : I \cdot R \rightarrow \text{Id} \). I claim it is an isomorphism.

Since the functor \( R \) is conservative it is enough to show
that its composition with the functor \( R \), i.e. morphism of functors \( R \cdot I \cdot R \rightarrow R \) is an isomorphism.

However, we know that the composition \( R \rightarrow R \cdot I \cdot R \rightarrow R \) is an identity morphism and looking at dimensions we see that the morphism \( R \cdot I \cdot R \rightarrow R \) is an isomorphism.

We have seen that not always we have a descent. Here is some criterion for the descent.

**Claim.** Suppose that for every \( d \) the cohomology group
\[ H^1_{\text{cont}}(H, \text{GL}(d, C)) \] is trivial. Then there is a descent from \( C \) to \( L \).

Now let us come back to situation when \( H = \text{Gal}(\bar{K}/K_\infty) \)
and show that in this case we have a descent. This result is due to Tate and Sen. It reduces the study of the cat-
egory $\text{Rep}(G, C)$ to the study of much simpler category $\text{Rep}(\Gamma, L)$. It is much simpler since the groups $\Gamma$ is almost isomorphic to $\mathbb{Z}_p$.

Remark on the proof of this result in the paper by Brion and Conrad.

Let us recall some things from cohomology theory.

9.1.2. Cohomology. 1. Discrete groups. $H^0(G, A), H^1(G, A)$.

If $A$ is a commutative group we can also define $H^i(G, A)$.

One of definitions to use cochain complex $0 \to C^0 \to C^1 \to \ldots$ where $C^i$ is the group of functions from $G^i$ to $A$.

**Theorem 9.2.** Let $M/L$ be a finite Galois extension of fields with the Galois group $G = \text{Gal}(M/L)$. Then

(i) $H^i(G, M^+) = 0$ for all $i > 0$.

(ii) $H^1(G, GL(d, M)) = 1$ (Hilbert 90)

9.2.1. Continuous cohomology. The same definition with continuous functions.
Let us discuss different levels of acyclicity. Suppose we have a complex $aA \to B \to C$ with morphisms $d, d'$. We say that it is acyclic at place $B$ if it satisfies the following condition

**Acyclicity 0.** $\text{Ker} d' = \text{Im} d$.

Now suppose that our groups are equipped with metrics and differentials $d, d'$ are continuous (i.e. bounded) morphisms. Then we can impose some stronger conditions

**Acyclicity 1.** There exists a constant $C > 0$ such that if $b \in B$ is a cycle then there exists an $a \in A$ such that $da = b$ and $\|A\| \leq C\|b\|$.

In fact, it is better to consider slightly stronger condition

**Acyclicity 2.** There exists a constant $C > 0$ such that for any $b \in B$ we can find an element $a \in A$ such that $H\|a\| \leq C\|b\|$ and $\|b - da\| \leq C\|d'b\|.$
Consider the situation as before. Let a profinite group $H$ continuously act on the field $C = C_K$. Let us set $L = C^H$.

**Theorem 9.3.** Suppose that the complex defining the continuous cohomology is strongly acyclic, i.e., it satisfies the condition Acyclicity 2 at $C^1$. Then $H^1(H, \text{GL}(d, C) = 1)$.

\[ H^1(\pi, \text{GL}(d, c)) = 0 \quad \text{for } c \in \text{Mat}(d, C). \]

Let $c$ be a cycle in $H^1(H, \text{GL}(d, C))$ central.

**Step 1.** Enough to find an open subgroup $H$ such that $c|_H$ is trivial.

$c$ corresponds to some open subgroup $G$ of the $\text{GL}(d, C)$.

$G$ is trivial.

So, I can decent to the field $K = C^G$ a finite extension of $C$. For finite extensions $K/L$, $\text{GL}(d, L)$.
c : R → C, cont.\n
choose small neighborhood \( U \) of \( 0 \).
\( E_U(d, \delta) \) and take
\( \mathcal{H}_0 = p^{-1}(U) \)

choose in \( C \) ideal \( \mathcal{O}_2 \)
\( \| \omega \| > p^{-1} \delta \), \( \forall \omega \in \mathcal{O}_2 \).
\( \mathcal{O}_2 = \{ \omega / \| \omega \| \leq 1 \} \).

consider ideal
\( \overline{\mathcal{O}} \subset \mathcal{O}_2 \), \( \mathcal{D} = \{ \omega / \| \omega \| \leq \frac{1}{2} \}
\)
\( \text{Mat}(d, \mathcal{D}) \).
\( U = 1 + \text{Mat}(d, \mathcal{D}) \subset \text{GL}(d, \mathcal{O}) \).
\( \mathcal{H}_0 = p^{-1}(U) \).

also \( \mathcal{O}_2 \) is trivial in coherent.

compare \( H^1(\mathcal{O}_2, U) \) and \( H^1(\mathcal{O}_0, \mathcal{M}) \),
\( \mathcal{M} = \text{Mat}(d, \mathcal{D}) \).
\( \mathcal{P} \simeq \mathcal{V} \) \( \mathcal{M} \simeq \mathcal{V} + \mathcal{W} \).
\( c : \mathcal{H}_0 \rightarrow \mathcal{V} \), \( c_1 \) is \( \mathcal{O}_2 \) - \( \mathcal{M} \).
\( c' = d \alpha + \varepsilon \), \( \varepsilon \) - smooth. \( \alpha \in \mathcal{O} \).
\( \alpha = c \), \( \mathcal{E} \subset \mathcal{L} \leq \frac{1}{2} \| \alpha \| \).
Let us see how to prove this stronger acyclicity condition 2 in our case when $L$ is the closure of the field $K_\infty$.

Given a cochain $c \in C^1(H, C)$ we can approximate it by a function $c'$ that is locally constant on $H$ and lies in $K$.

Hence we can assume that there exists a subgroup $H_0$ of finite index in $H$ that corresponds to a finite field extension $M/L$ such that our cocycle reduces to a cocycle $c' \in C^1(H', M)$, where $H' = H/H_0$ is a finite group.

We should just check that for all these finite groups we can choose the same constant $C$ in condition Acyclicity 2.

This would follow from the following theorem due to Tate.

**Theorem 9.4.** For any finite extension $M/L$ we have $tr(\mathcal{O}_M)$ contains the maximal ideal $\mathfrak{m}$ of the ring $\mathcal{O}_L$.

We will prove this later.
Now let us recall how to prove the acyclicity of the cohomology $H^1(Gal(M/L), M)$.

Reminder from cohomology theory. Let $(C, d)$ be a complex of abelian groups.

**Homotopy** is an operator $D : C \rightarrow C$ of degree $-1$. Such homotopy induces an endomorphism $\nu_D$ of the complex $C$ via $\nu_D = dD + Dd$.

Morphism $\nu_D$ induces zero morphism on cohomologies. Thus, if this morphism is identity (or is invertible) this would guaranty acyclicity. In fact this is the standard way to prove acyclicity.

Let $M/L$ be a finite Galois extension. Let us recall how to prove that $H^i(G, M) = 0$ for $i > 0$. Choose an element $m \in M$ such that $tr(m) = 1$. Such an element defines a homotopy $D = D_m$. On 1 cochains it is given by $\sum c_ig_i \mapsto \sum c_ig_i(m)$. It is clear that $\nu_D = Id$ that implies acyclicity.

Now come back to our situation. Let $L \subset C$ as before, $M \subset C$ a finite Galois extension of $L$.

By Tate theorem we can choose an element $m \in M$...
such that $m = 1$ and $||m|| < 2$. Then the corresponding homotopy $D_m$ has norm $\leq 2$ and hence Acyclicity 2 condition holds with the constant $c = 2$.

\[ H^1(G, GL(d, \mathbb{C})) = C^1 \text{ for all } \]

\[ \text{Dense } \text{Rep} (G, \mathbb{C}) = \text{Rep} (G, \mathbb{C}). \]

Is not reduced to

\[ \text{Var}(K) \neq \text{Var}(K') \neq \text{Rep}(K') \]

Obstruction is given by sen operator $D_s$.

Interesting to consider $W$-vector space over $\mathbb{C}$ and $D_s: V \to V$.

\[ \Phi: \text{End}(V, W) \to \text{Rep}(W, W) \]

\[ \text{Leaves} \text{W defines Cartan}

\[ \mathfrak{g} \text{ - Comparison between product fields and field in char p.} \]

\[ \text{K - algebraic volunter fixed, be newvalue fixed, then} \]

\[ K \text{ trans similar to } K \]