2. Lecture 2. Betti cohomology, etale cohomology and l-adic cohomology.

2.1. Etale cohomology. Let X be an algebraic variety over an algebraically closed field L. In 1940-th A. Weil conjectured that in some cases one can define **in purely algebraic terms** the cohomology groups $H^*(X, A)$ that have good functorial properties and in case when $L = \mathbb{C}$ would coincide with the groups $H^*(X_{an}, A)$.

This program has been realized by A. Grothendieck. Namely, given a finite abelian group A such that the size $\#(A)$ is prime to the $p = char(L)$ he defined the etale cohomology groups $H_{et}(X, A)$ that have all the desired properties.

2.2. l-adic cohomology. Using this construction Grothendieck defined a family of cohomology theories for such varieties. Namely , for every prime number l prime to $char(L)$ he defined l-adic cohomology functors

$$H^*(X, \mathbb{Z}_l) := \lim H^*(X, \mathbb{Z}/l^k\mathbb{Z})$$
$$H^(X, \mathbb{Z}_l) := \mathbb{Q}_l \otimes_{\mathbb{Z}_l} H^*(X, \mathbb{Z}_l)$$
$$H^*(X, \bar{\mathbb{Q}}_l) := \bar{\mathbb{Q}}_l \otimes_{\mathbb{Q}_l} H^*(X, \mathbb{Q}_l).$$

In fact Grothendieck and his group has done much more.

Constructible sheaves, Six functors etc.

2.3. Cohomology of elliptic curves. I will use these cohomology theories and their properties without proofs. But in order to show what are the difficulties and limitations of these theories I will consider a special case where all these theories could be described much more explicitly.

Example 2.3.1. Consider an elliptic curve E over an algebraically closed field L. The set $E(L)$ of L-points of E has a natural structure of an abelian group.

For every natural number n consider the multiplication morphism $[n] : E \to E$. This morphism is epimorphic, and we denote by $E(n) \subset E(L)$ the kernel of this morphism.

2.3.2. Consider the case when $L = \mathbb{C}$. In this case we have an analytic realization E_{an} of our curve as $E_{an} = \mathbb{C}/\Lambda$, where Λ is some lattice in $\mathbb{C}$. From this realization can easily describe all topological invariants of the space E_{an}.

Claim. *(i)* $H_1(E_{an}, \mathbb{Z}) = \Lambda,\ H^1(E_{an}, \mathbb{Z}) = \Lambda^*$

(ii) $E(\infty) = \mathbb{Q}/\mathbb{Z} \otimes \Lambda$ *and for every* n *we have* $E(n) = \frac{1}{n}\Lambda/\Lambda \approx (\mathbb{Z}/n\mathbb{Z})^2$

All information about the lattice Λ that we can extract from the algebraic curve E is encoded into the group $E(\infty)$. It allows to reconstruct many objects related to the group $\Lambda = H_1(E_{an}, \mathbb{Z})$, but not Λ itself. Here are some of constructions.

(i) Let l be a prime number. Consider the $\mathbb{Z}_l$-module $T_l(E) := Mor(\mathbb{Q}_l/\mathbb{Z}_l, E(\infty)$. Then it is equal to the l-adic completion of the lattice Λ, i.e. to the $\mathbb{Z}_l$-module $\mathbb{Z}_l \otimes \Lambda$.

(ii) The $\mathbb{Q}_l$-module $T_l' = \mathbb{Q}_l \otimes_{\mathbb{Z}} \Lambda$ can be described as the group of continuous morphisms

$Mor_{con}(\mathbb{Q}_l, E(L)) = Mor_{con}(\mathbb{Q}_l, E(\infty))$, where $E(L)$ is considered with discrete topology.

2.3.3. From Lefschetz principle we can see that we can prove similar results for any algebraically closed field L of characteristic 0. Namely, we claim that

(i) $E(n) \approx (\mathbb{Z}/n\mathbb{Z})^2$

(ii) The $\mathbb{Z}_l$ module $T_l(E) : Mor(\mathbb{Q}_l/\mathbb{Z}_l, E(L)$ is isomorphic to ${\mathbb{Z}_l}^2$

(iii) The module $T_l'(E) = Mor_{con}(\mathbb{Q}_l, E(L))$ is isomorphic to ${\mathbb{Q}_l}^2$.

In case when L is an algebraically closed field of characteristic $p > 0$ the same results hold provided n and l are prime to p. They are not correct if this does not hold.

This explains what are limitations of these cohomology theories and why they behave badly if l equals to the characteristic of L.

2.4. Action of Galois groups. Main advantage o the theories defined by purely algebraic constructions is that they produce representations of Galois groups. Namely, suppose we are given a field K. Fix an algebraic closure $L = \bar{K}$ and denote by Γ the Galois group $\Gamma = Gal(L?K) = Aut_K(L)$.

Suppose we are given an algebraic variety Y defined over K. Let us denote by $X = Y_L$ the variety over L obtained from Y by extension of scalars. Then the Galois group Γ acts on the scheme (variety) X and this induces the action of Γ on all cohomology spaces associated to X – $H_{et}(X, A), H(X, \mathbb{Z}_l), H(X, \mathbb{Q}_l$. Thus, starting from the variety Y we produce families of representations of the group Γ. We usually will denote these representations by ρ or ρ_l.

Remarks. 1. This construction gives a way to construct interesting families of Galois representations. In fact, this gives a powerful tool to study Galois groups – they are some of the most important characters in Number Theory.

2. Starting with the variety Y we constructed the whole series of representations $(\rho_l, \Gamma, H^i(X, \mathbb{Q}_l)$. If you look at their construction, even in the simple case when Y is an elliptic curve defined over K, you do not see any direct relations between these representations.

On the other hand one feels that in some sense this is the same representation – in slightly different disguises. One can suspect that in fact there exists some representation ρ in vector space over $\mathbb{Q}$ that produces these representations (at least for almost primes l).

This suggestion is probably not correct as stated. However, I personally think that there exists some "small" object ρ that allows to reconstruct these representations ρ_l for almost all l.

2.4.1. *Case of p-adic fields.* Let K be a number field. Consider its non-Archimedean valuation $\mathfrak{p}$ and denote by $K_{\mathfrak{p}}$ the completin of K with respect to this valuation.

Here we can think about the case when $K = \mathbb{Q}, K_{\mathfrak{p}} = \mathbb{Q}_p$.

It is known that the absolute Galois group $\Gamma_{\mathfrak{p}}$ can be naturally embedded into Γ (this

Let as before $L = \bar{K}$ and $\Gamma = Gal)L/K)$. Consider the family of representations ρ_l of the type describe above.

In order to study a representation ρ of the global Galois group Γ it is usually useful to first study its restriction to a local Galois group $\Gamma_{\mathfrak{p}}$. So, let us consider a representation ρ_l restricted to the local Galois subgroup $\Gamma_{\mathfrak{p}}$.

We will see that in case when l is not equal to the residual characteristic p of the field $K_{\mathfrak{p}}$ the structure of any continuous representation of the Galois group $\Gamma_{\mathfrak{p}}$ is relatively simple. Thus the most difficult and interesting case is when $l = p$. This is exactly the subject of the p-adic Hodge theory.

2.5. Structure of the local Galois group and its representations. In this course we will mostly deal with local Galois groups.

Let us fix a p-adic field K that is a finite extension of the field $\mathbb{Q}_p$. W fix a discrete valuation $v : K \to \mathbb{Q}bigcup\infty$ normalized by condition $v(p) = 1$. We denote by $O = O_K$ the ring of integers, $O_K = \{x \in K | v(x) \geq 0\}$. Let $\mathfrak{p} = \mathfrak{p}_+K; = \{x \in O_K | v(x) > 0\}$ be the maximal ideal of K and $k = O_K/\mathfrak{p}$ the finite residue field.

Fix the algebraic closure L/K and denote by Gam the Galois group $Gal(L/K$. We extend the valuation v to the field L (this is not a discrete valuation). Using the valuation v we define the ring of integers $O_L \subset L$ – this is the integral closure of the ring O_K in L.

The ring O_L is a local ring and the residue quotient field $l = O_L/\mathfrak{p}_L$ is an algebraic closure of the residue field k.

From this construction we see that there is a canonical morphism $p : Gal(L/K) \to Gal)l/k)$. It is known that this morphism is epimorphic.

The group $Gal(l/k)$ is isomorphic to the group $\hat(\mathbb{Z})$. In fact this isomorphism is canonical since this group has the distinguished element – Frobenius morphism – $FR = Fr_q$, where $q := \#(k)$. This element plays a central role in the theory.

The kernel of the morphism $p : Gal(L/K) \to Gal(l/k)$ is called the **inertia group** of K. We denote this subgroup by I_K.

Definition. A representation ρ of the group $Gal(L/K)$ is called **unramified** if it is trivial on the inertia subgroup I_K.

An unramified representation (ρ, V) can be considered as a representation of the quotient group $Gal(l/k)$. In particular it defines an automorphism $Fr : V \to V$ equal to $\rho(Fr_q)$.

Let Y be a smooth projective variety defined over the field K. We say that Y has a **good reduction** if it can be realized over the ring O_K in such a way that the quotient variety $\bar{Y}$ over the field k is again smooth and projective.

Theorem 2.6. *Suppose that a variety Y over the field K has good reduction. Fix a prime number l different from the characteristic p of the field k.*

Then the representation $(\rho_l, Gal(L/K), H^*(X, \mathbb{Q}_l))$ of the group $Gal(L/K)$ is unramified.

Moreover, there exists a canonical isomorphism of vector spaces $H^*(X, \mathbb{Q}_l)$ and $H^*(\bar{X}.\mathbb{Q}_l)$ compatible with the action of the Galois group $Gal(L/K)$.

In case when when $Y = E$ is an elliptic curve this follows from the statement the the reduction map $E(n) \to \bar{E}(n)$ is bijective for any n prime to p.

char $k = p$

$\Gamma = Gal(L/K)$ acts on p-adic space

$H^2(X, \mathbb{Q}_p)$.

Structure of $Gal(L/K)$, K-p-adic field.

$Gal(L/k)$

$$1 \to I_K \to Gal(L/K) \to \hat{\mathbb{Z}} \to 1$$

Group I_K corresponds to the maximal unramified field extension $K \subset K^{un} \subset L$

$I_K = Gal(L/K^{un})$.

$K^{un} = K(\sqrt[n]{1})$, for all n prime to p.

$K^{tr} = K^{un}(\sqrt[n]{x})$ | n-prime to p

$(K^{un})^* / (\mathcal{O}^{un})^* = \mathbb{Z}$

corresponding sgr. of Γ is denoted W_k - wild ramification group.

Lemma $I_k / W_k \cong \hat{\mathbb{Z}}^{\neq p} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$

claim: W_p is a p-group.

structure of $\Gamma = Gal(\bar{k}/k)$

$\Gamma \supset I_k \supset W_k$ normal sgr.

$\Gamma / I_k \cong \hat{\mathbb{Z}}$, generated by Fr

$I_k / W_k \cong \hat{\mathbb{Z}}^{\neq p}$

W is a p-group.

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corollary. Let $\ell \neq p$. Then for any continuous rep.

$\rho : \Gamma \to End(V)$ of some f.dim ℓ-adic vector space.

we have $\rho(W_k)$ is finite.

$\rho : \Gamma \to GL(V)$

Γ / W_k

If $\bar{\ell} = p$ then p-adic repr. of Γ are quite non-trivial since they are quite non-trivial on U_K.

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cyclotomic character.

$N = K(\sqrt[p^k]{1}) \subseteq L = \bar{K}$

Γ acts on N, i.e. acts on ~~the~~ group $R = \{\sqrt[p^k]{1}\}$ of $\sqrt[p^k]{1}$

$R = (\mathbb{Q}_p/\mathbb{Z}_p) =$

$R = G_m(p^\infty) \subset L^*$.

Γ acts on R.

$T(R) = \mathrm{Mor}(\mathbb{Q}_p/\mathbb{Z}_p, L^*) =$

$= \mathrm{Mor}(\mathbb{Q}_p/\mathbb{Z}_p, R)$

Γ acts on $T(R)$, $T(R) \simeq \mathbb{Z}_p$

1-dim. repr. of Γ over $\mathbb{Z}_p$

Given by $\rho(\gamma) = \chi(\gamma)$

$\rho(\gamma)(x) = x^{\chi(\gamma)}$

$x \in D$

$x \in P$, $\alpha \in \mathbb{Z}_p$ then we can define x^{α}.