4. Lecture 4. Some rings of periods

4.1. \textit{$p$-adic fields.} We fixed a prime number $p$. Consider a complete valued field $K$ of characteristic 0 with valuation $v : K \to \mathbb{Q} \cap \infty$ such that $v(p) = 1$.

Let $O_K$ denote the ring of integers in $K$, $p_K$ its maximal ideal and $k = O_K/p_K$-its residue field of characteristic $p$.

We are mostly interested in the case when $K$ is a finite extension of $\mathbb{Q}_p$. However, sometimes it is convenient to extend the residue field $k$ to its algebraic closure. So we adopt the following terminology.

\textbf{Definition.} A \textit{$p$-adic field} is a field $K$ of characteristic 0 with a discrete valuation $v : K \to \mathbb{Q} \cup \infty$ such that $K$ is complete with respect to $v$ and its residue field $k$ is a perfect field of characteristic $p$.

For $p$-adic field $K$ we choose a uniformizer $\pi = \pi_K$, i.e. any generator of $p_K$ as $O_K$-module

\textit{Examples.} 1. Any finite extension $K$ of $\mathbb{Q}_p$ is a $p$-adic field.

2. Let $K$ be a $p$-adic field. Fix an algebraic closure of $K$ and consider the maximal unramified extension $L = K^{un}$ of $K$. This field has a discrete valuation $v$ and its residue field $l$ is isomorphic to the algebraic closure of the residue field $k$ of $K$. 
However, the field $L$ is not complete with respect to $v$, so it is not a $p$-adic field. The completion $L'$ of $L$ with respect to $v$ is a $p$-adic field with residue field $l = \bar{k}$.

**Exercise.** Let $L$ be an algebraic field extension of a $p$-adic field $K$. Then the valuation $v$ uniquely extends to a valuation of the field $L$. If $L/K$ is a finite extension then $L$ is a $p$-adic field with this valuation.

**Teichmüller representatives.**

$p$-adic field,

$0_{L} \xrightarrow{v} k = \mathcal{O}_{K}/(\mathfrak{p}_{K})$

Claim: $s_{v}(x_{1}) - s_{v}(x_{1}) \in \mathfrak{m}$

Claim: $s_{v}(x_{i}) = s_{v}(x_{i+1})$

Claim: $v(x) = v(y^{p})$

$\sum (\mathfrak{m}^{i})^{p} = \mathfrak{m}(\mathfrak{m} + \mathfrak{m})^{p}$

$\sum y_{i}^{p} = \sum y_{i} + \sum y_{i}^{p}$

Define $e_{n}(x)$

$y^{p^{n}} = x$

$-\text{c}e^{x}$ if $y$ defines $w_{0}$

$w_{0}$
Choose uniform $\tau = \varepsilon_u$.

Proposition: Every element $x \neq 0$ can be uniquely written

$$\sum_{\mu \geq 0} s(\omega_1 \tau^u \varepsilon_u)$$

$\varepsilon_u = 0$ for $u < 0$

Let $\mathbb{I}$ be a topel group.

$B$ is $\mathbb{Q}_p$-vector.

and $B$ is $\varepsilon$-module over $\mathbb{Q}_p$.

$\text{Rep}_B(\mathbb{I})$ = finite group $\mathbb{R}$-mod

with convolution $\ast$

$\varepsilon(B) = \varepsilon(B)$ for $u < 0$.

$\mathbb{I}$-modules.
4.1.1. Teichmuller representatives. ♠

Proposition 4.1.2. Let $K$ be a $p$-adic field, $O_K$ its ring of integers and $p: O_K \rightarrow k$ the residue map. Then there exists unique multiplicative section $s: k \rightarrow O_K$.

Element $s(a)$ is called the Teichmuller representative of an element $a \in k$.

Exercise. ♠

Let $\pi \in \mathfrak{p}_K$ be a uniformizer. Then any element $x \in K$ can be uniquely written as

$$x = \sum_{i \in \mathbb{Z}} s(a_i) \pi^i$$

where $a_i \in k$ and $a_i = 0$ for $i \ll 0$. 

\[ \text{Claim. Category has tensor products.} \]

\[ M \otimes_b N \text{ is a } (B, B)\text{-module.} \]

If $B$ is a field, there exists a unique field homomorphism

$$M \otimes_b \mathbb{F}_\pi \rightarrow B$$

\[ \text{Fontaine. } K\text{-p-adic field, } \Gamma = \text{Gal } (\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \text{.} \]
Ring $\mathcal{O}_E$ is a $B$-algebra of periods, with action of $E$.

Assume that $\mathcal{O}_E$ is a $p$-adic field.

Then we have a function

$$D_\mathcal{O} : \text{Rep}_{\mathcal{O}_E}(M) \to \text{Vec}_k$$

$$M \to B \otimes_{\mathcal{O}_E} M \to (B \otimes_{\mathcal{O}_E} M)$$

**Def.** $M$ is called $B$-admissible.

$D_\mathcal{O}$-admissible if

$$\dim_{\mathcal{O}_E} M = \dim_{\mathcal{O}_E} \text{Rep}(M)$$

Claim. If $B$ is a field then always $\dim D_\mathcal{O}(M) \leq \dim M$

$$\dim D_\mathcal{O}(M) \leq \dim M$$

Example.

1. $B = \mathcal{O}_E$

Claim. $M$ is $B$-admissible if and only if it is smooth.
\[ M = \mathbb{C}, \mathbb{Q} \text{-module} \]

\[ B \otimes M = \mathbb{C} \otimes (\mathbb{Q} \otimes M) \]

**Example.** \( \mathbb{C}k < N \) finite dimensional \( \mathbb{C}k \)-module.

\[ \text{Prop.} \quad \text{Rep}(k) \rightarrow \text{Vect}, \quad \Gamma = \text{alg} \]

**Prop.** \( \text{any } \Gamma \text{-module } M \) is admissible.

\[ M \rightarrow M' = \mathbb{C} \otimes M \rightarrow \mathbb{Q} \otimes \mathbb{C} \otimes M' \rightarrow \mathbb{Q} \otimes M' \]

Can argue \( \mathbb{Q}k \)

\[ M = \mathbb{C}, \mathbb{Q} \text{-module, } \Gamma \text{tors} \]

\[ \dim(\mathbb{W} \otimes \mathbb{M}) = \dim M \]

**Prop.** \( M \) - vector space over \( \mathbb{K} \)

\( M \) has automorphisms

\( M \) has about \( F \)

choose a large field

extension \( \mathbb{K} \rightarrow N, N \supset \mathbb{K} \)

\( (\mathbb{F}, N) \text{-module } \mathbb{F} = N \otimes \mathbb{M} \)

extension \( (\mathbb{F}, N) \text{-module } \mathbb{F} \)

\[ \mathbb{F} \text{ factors in } \Gamma \]
\[ N' = L \]

Claim: \[ \text{Rep}_p(L) \leq V_* \]

Shrinking \( m \) over \( L \)

\[ \text{Mav}(N, L) = D_1 \text{ is } \Gamma \text{-comm.} \]

\[ L_m = L \]

\[ n = L_m \text{ is alg.} \]

Frob. \( \text{Rep}_p(N) \subseteq \text{Rep}_p(L) \Rightarrow \text{odd} \]

\[ (= \text{odd}(p^k)) \]

If \( N \) is smooth.

Reduce to finite field.

\[ N \leq \text{Ser. of degree} \]

st. \( D \) odd and \( \Gamma \) is even.

\[ N \rightarrow \Phi(N) = \text{ker}(N, D) \]

\[ N' = N'' \]

\[ D = \overline{D} \]

If \( N' \) is not smooth.

it is passive.

smooth quotient of

\[ D_{\overline{D}} \rightarrow N' \rightarrow \text{Q}(N) \]

\[ D(N) = \text{Hom}_w(N', \overline{D}) \]

\[ w : N' \rightarrow \overline{D} \]
Period ring \( C = C_k \).

\[
\dim \mathcal{D}(M) \leq \dim \text{Hom}(O(M), M) \leq \dim O(M).
\]

\( \mathcal{D}(M) \) is a \( \mathcal{O}(M) \)-module.

**Proposition.** \( C \) is algebraically closed.

1. \( C = \overline{K} \) is algebraically closed.

2. Every \( \mathcal{D}(M) \) is algebraically closed.

\( \mathcal{D}(M) \) is a \( \mathcal{O}(M) \)-module.

**Theorem.**

Let \( X \) be smooth over \( K \).

- \( X_k \) is \( \mathcal{O}(M) \)-module.

\( M \) is \( \mathcal{O}(M) \)-module.

\[
\mathcal{D}_o(M) = \mathcal{O}(M) \otimes_{\mathcal{O}(M)} K^o(\ell, \mathfrak{A}, \text{prev} = \alpha)
\]

**Adeletoide chain.**
K - algebraic field.
F = Gal(K/k)

There exists a normal 1-dim. resol of K over Kp.

Lemma: \( \mathbb{Q}_p / \mathbb{Q}_p \cdot \mathbb{Z}_p \) - abelian group.

Then \( \text{ker } (R_p, R_p) = \mathbb{Z}_p \)
\( p = \mathfrak{p} \cdot m(k) \), \( p(R) = \mathfrak{a} \).

Denote \( T_p(\mathfrak{p} \cdot m(k)) = \text{Mor}(R_p, \mathfrak{p} \cdot m(\mathfrak{p})) = \text{inn} \).

The module \( (T, \mathfrak{a}_p) \)-module.

\( T(\mathfrak{a} \cdot m(k)) = \mathfrak{a} \cdot m(\mathfrak{a}) \cdot \mu_p(\mathfrak{a}) \).

\[ T = H^2_{\text{et}}(\mathfrak{p} \cdot m(k), \mathbb{Z}_p) \]
\( T \text{ mod } = \text{ker } (\text{ker}(\mathfrak{a} \cdot m(\mathfrak{a}), \text{ker}(\mathfrak{a})) \).

\( \text{Der}_{\mathbb{Q}_p}(k) \).

\[ T_p = (\mathfrak{a}_p, \mathfrak{p}_n) \quad \mathfrak{a}_p = \mathfrak{a}^n \]

\( T_p = T_p \cdot \mathfrak{p} \cdots \mathfrak{p} \mathfrak{p} \quad \mathfrak{p}_n > \mathfrak{p} \)
\( T_p = T_{\mathfrak{p}_n} = \mathfrak{a}^n \mathfrak{a}^n \quad n > 0 \)
\( T_p = T_{\mathfrak{p}_n} = \mathfrak{a}^n \mathfrak{a}^n \quad n < 0 \)

\[ T_p(\mathfrak{a} \cdot m(k)) = \text{Mor}(R_p, \mathfrak{p} \cdot m(\mathfrak{a}), \mathfrak{a}^n). \]
\( \text{Mor}(R_p, \mathfrak{p} \cdot m(\mathfrak{a}), \mathfrak{a}^n) \)
\( T_{\mathfrak{p}_n} = \mathfrak{p}_n \).