VANISHING OF CERTAIN EQUIVARIANT DISTRIBUTIONS ON p-ADIC SPHERICAL SPACES, AND NON-VANISHING OF SPHERICAL BESSEL FUNCTIONS

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Abstract. We prove vanishing of distribution on p-adic spherical spaces that are equivariant with respect to a generic character of the nilradical of a Borel subgroup and satisfy a certain condition on the wave-front set. We deduce from this non-vanishing of spherical Bessel functions for Galois symmetric pairs.

1. Introduction

Let $G$ be a reductive group, quasi-split over a non-Archimedean local field $F$ of characteristic zero. Let $B$ be a Borel subgroup of $G$, and let $U$ be the unipotent radical of $B$. Let $H$ be a closed subgroup of $G$. Let $G,B,U,H$ denote the $F$-points of $G$, $B$, $U$, $H$ respectively. Suppose that $H$ is a $F$-spherical subgroup of $G$, i.e. there are finitely many $B \times H$-double cosets in $G$. Let $g, h$ be the Lie algebras of $G, H$ respectively. Let $\psi$ be a non-degenerate character of $U$ and let $\chi$ be a (locally constant) character of $H$. For $x \in G$ denote $H \cdot x := xHx^{-1}$ and denote by $\chi^x$ the character of $H^x$ defined by conjugation of $\chi$. For a $B \times H$-double coset $O \subseteq G$ define

$$O_c := \{ x \in O \mid \psi|_{H^x \cap U} = \chi^x|_{H^x \cap U} \}.$$

Let

$$Z := \bigcup_{O \neq O_c} O.$$

Identify $T^*G$ with $G \times g^*$ and let $N^*_g$ be the set of nilpotent elements in $g^*$.

Consider the action of $U \times H$ on $G$ given by $(u, h)x = uhx^{-1}$. This gives rise to an action of $U \times H$ on the space $S(G)$ of Schwartz (i.e. locally constant compactly supported) functions on $G$ and the dual action on the space of distributions $S^*(G)$.

In this paper we prove the following theorem.

Theorem A (see Section 3). Let $\xi \in S^*(G)^{(U \times H, \psi \times \chi)}$ be an equivariant distribution on $G$, i.e. $(u, h)\xi = \psi(u)\chi(h)\xi$. Suppose that the wave-front set (see section 2.2) $WF(\xi)$ lies in $G \times N^*_g$, and $\text{Supp}(\xi) \subseteq Z$. Then $\xi = 0$.

In the case when $H$ is a subgroup of Galois type we can prove a stronger statement. By a subgroup of Galois type we mean a subgroup $H \subset G$ such that

$$(G \times_{\text{Spec}F} \text{Spec}E, H \times_{\text{Spec}F} \text{Spec}E) \simeq (H \times_{\text{Spec}F} H \times_{\text{Spec}F} \text{Spec}E, \Delta H \times_{\text{Spec}F} \text{Spec}E).$$
Corollary B (see Section 4). Let \( H \subset G \) be a subgroup of Galois type, and let \( \chi \) be a character of \( H \). Let \( S \) be the union of all non-open \( B \times H \)-double cosets in \( G \). Let \( \xi \in S^*(G)^{(U \times H, \psi \times \chi)} \). Suppose that \( WF(\xi) \subset G \times N_{g^*} \) and \( \text{Supp}(\xi) \subset S \). Then \( \xi = 0 \).

Note that if \( \chi \) is trivial, we can consider the distribution \( \xi \) as a distribution on \( G/H \). Considering \( \tilde{G} := G \times G \) and taking \( H \) to be the diagonal copy of \( G \) we obtain the following corollary for the group case.

Corollary C (see Section 4). Let \( \psi_1 \) and \( \psi_2 \) be non-degenerate characters of \( U \). Let \( B \times B \) act on \( G \) by \((b_1, b_2)g := b_1gb_2^{-1}\). Let \( S \) be the complement to the open \( B \times B \)-orbit in \( G \). For any \( x \in G \), identify \( T_xG \) with \( g \) and \( T_x^*G \) with \( g^* \). Let

\[ \xi \in S^*(G)^{(U \times U, \psi_1 \times \psi_2)} \]

and suppose that \( WF(\xi) \subset S \times N_{g^*} \). Then \( \xi = 0 \).

1.1. Applications to non-vanishing of spherical Bessel functions. Let \( \pi \) be an admissible representation of \( G \) (of finite length), and \( \tilde{\pi} \) be the smooth contragredient representation. Let \( H \subset G \) be an algebraic spherical subgroup and let \( \chi \) be a character of \( H \). Let \( \phi \in (\pi^*)^{(U, \psi)} \) be a \((U, \psi)\)-equivariant functional on \( \pi \) and \( v \) be an \((H, \chi)\)-equivariant functional on \( \tilde{\pi} \). For any function \( f \in S(G) \), we have \( \pi^*(f)\phi \in \tilde{\pi} \subset \pi^* \).

This enables us to define the spherical Bessel distribution corresponding to \( v \) and \( \phi \) by

\[ \xi_{v, \phi}(f) := \langle v, \pi^*(f)\phi \rangle. \]

By [AGS, Theorem A] we have \( WF(\xi_{v, \phi}) \subset G \times N_{g^*} \).

The spherical Bessel function is defined to be the restriction \( j_{v, \phi} := \xi_{v, \phi}|_{G \times S} \), where \( S \) is the union of all non-open \( B \times H \)-double cosets in \( G \). One can easily deduce from [AGS, Theorem A] and Lemma 3.1 that \( j_{v, \phi} \) is a smooth function. Theorem A and Corollary D imply the following corollary.

Corollary D. Suppose that \( \pi \) is irreducible and \( v, \phi \) are non-zero. Then

(i) For any open subset \( U \subset G \) that includes \( G \setminus Z \) we have \( \xi_{v, \phi}|_{U} \neq 0 \).

(ii) If \( H \) is a subgroup of Galois type then \( j_{v, \phi} \neq 0 \).

For the group case this corollary was proven in [LM, Appendix A].

1.2. Related results. In [AG] a certain Archimedean analog of Theorem A is proven (see [AG, Theorem A]). This analog implies that the Archimedean analog of Corollary D(ii) holds for any spherical pair \((G, H)\) (see [AG, Corollary B]).

Corollary C together with [AGS, Theorem A] can replace [GK75, Theorem 3] in the proof of uniqueness of Whittaker models [GK75, Theorem C].

Theorem A can be used in order to study the dimensions of the spaces of \( H \)-invariant functionals on irreducible generic representations of \( G \) (see [AG, §1.3] for more details). It can also be used in the study of analogs of Harish-Chandra's density theorem (see [AGS, §1.7] for more details).

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2. Preliminaries

2.1. Conventions.

- We fix $F, G, B, U, X$ and $\psi$ as in the introduction.
- All the algebraic groups and algebraic varieties that we consider are defined over $F$. We will use capital bold letters, e.g. $G, X$ to denote algebraic groups and varieties defined over $F$, and their non-bold versions to denote the $F$-points of these varieties, considered as $l$-spaces or $F$-analytic manifolds (in the sense of [Ser64]).
- When we use a capital Latin letter to denote an $F$-analytic group or an algebraic group, we use the corresponding Gothic letter to denote its Lie algebra.
- We denote by $G_x$ the stabilizer of $x$ and by $g_x$ its Lie algebra.
- For an $F$-analytic manifold $X$, a submanifold $Y \subset X$ and a point $y \in Y$ we denote by $CN^X_Y \subset T^*X$ the conormal bundle to $Y$ in $X$, and by $CN^X_{Y,w}$ the conormal space at $y$ to $Y$ in $X$.
- By a smooth measure on an $F$-analytic manifold we mean a measure which in a neighborhood of any point coincides (in some local coordinates centered at the origin) with some Haar measure on a closed ball centered at $0$. A Schwartz measure is a compactly supported smooth measure.
- The space of generalized functions $G(X)$ on an $F$-analytic manifold $X$ is defined to be the dual of the space of Schwartz measures. One can identify $G(X)$ with $S^*(X)$ by choosing a smooth measure with full support.
- Let $\phi : X \to Y$ be a submersion of analytic manifolds. Note that the pushforward of a Schwartz measure with respect to $\phi$ is a Schwartz measure. By dualizing the pushforward map we define the pullback map $\phi^* : G(Y) \to G(X)$.

2.2. Wave front set. In this section we give an overview of the theory of the wave front set as developed by D. Heifetz [Hei85], following L. Hörmander (see [Hör90, §8]). For simplicity we ignore here the difference between distributions and generalized functions.

Definition 2.1.

1. Let $V$ be a finite-dimensional vector space over $F$. Let $f \in C^\infty(V^*)$ and $w_0 \in V^*$. We say that $f$ vanishes asymptotically in the direction of $w_0$ if there exists $\rho \in \mathcal{S}(V^*)$ with $\rho(w_0) \neq 0$ such that the function $\phi \in C^\infty(V^* \times F)$ defined by $\phi(w, \lambda) := f(\lambda w) \cdot \rho(w)$ is compactly supported.

2. Let $U \subset V$ be an open set and $\xi \in \mathcal{S}^*(U)$. Let $x_0 \in U$ and $w_0 \in V^*$. We say that $\xi$ is smooth at $(x_0, w_0)$ if there exists a compactly supported non-negative function $\rho \in \mathcal{S}(V)$ with $\rho(x_0) \neq 0$ such that the Fourier transform $\mathcal{F}^*(\rho \cdot \xi)$ vanishes asymptotically in the direction of $w_0$.

3. The complement in $T^*U$ of the set of smooth pairs $(x_0, w_0)$ of $\xi$ is called the wave front set of $\xi$ and denoted by $WF(\xi)$.

4. For a point $x \in U$ we denote $WF_x(\xi) := WF(\xi) \cap T^*_x U$.

Remark 2.2.

1. Heifetz defined $WF_\Lambda(\xi)$ for any open subgroup $\Lambda$ of $F^\times$ of finite index. Our definition above differs slightly from the definition in [Hei85]. They relate by $WF(\xi) - (U \times \{0\}) = WF_{F^\times}(\xi)$. 
(2) Though the notion of Fourier transform depends on a choice of a non-degenerate additive character of \( F \), this dependence effects the Fourier transform only by dilution, and thus does not change our notion of wave front set.

**Proposition 2.3** (see [Hor90, Theorem 8.2.4] and [Het85, Theorem 2.8]). Let \( U \subset F^m \) and \( V \subset F^n \) be open subsets, and suppose that \( \phi : U \rightarrow V \) is an analytic submersion. Then for any \( \xi \in S^*(V) \), we have
\[
WF(\phi^*(\xi)) \subset \phi^*(WF(\xi)) := \{(x, v) \in T^*U \mid \exists w \in WF_{\phi(x)}(\xi) \text{ s.t. } d^*_\phi(x)\phi(w) = v\}.
\]

**Corollary 2.4.** Under the assumption of Proposition 2.3 we have
\[
WF(\phi^*(\xi)) = \phi^*(WF(\xi)).
\]

**Proof.** The case when \( \phi \) is an analytic diffeomorphism follows immediately from Proposition 2.3. This implies the case of open embedding. It is left to prove the case of linear projection \( \phi : F^{n+m} \rightarrow F^m \). In this case the assertion follows from the fact that \( \phi^*(\xi) = \xi \boxtimes 1_{F^m} \) where \( 1_{F^m} \) is the constant function 1 on \( F^m \).

This corollary enables to define the wave front set of any distribution on an \( F \)-analytic manifold, as a subset of the cotangent bundle. The precise definition follows.

**Definition 2.5.** Let \( X \) be an \( F \)-analytic manifold and \( \xi \in S^*(X) \). We define the wave front set \( WF(\xi) \) as the set of all \( (x, \lambda) \in T^*X \) which lie in the wave front set of \( \xi \) in some local coordinates. In other words, \( (x, \lambda) \in WF(\xi) \) if there exist open subsets \( U \subset X \) and \( V \subset F^n \), an analytic diffeomorphism \( \phi : U \simeq V \) and \( (y, \beta) \in T^*V \) such that \( x \in U \), \( \phi(x) = y \), \( d\phi(\beta) = \lambda \), and \( (y, \beta) \in WF((\phi^{-1})^*(\xi|_V)) \).

**Theorem 2.6.** (Corollary from [A13, Theorem 4.1.5]) Let an \( F \)-analytic group \( H \) act on an \( F \)-analytic manifold \( Y \) and let \( \chi \) be a character of \( H \). Let \( \xi \in S^*(Y)^{(H,\chi)} \). Then
\[
WF(\xi) \subset \{(x, v) \in T^*Y | v(T_x(Hx)) = 0\}.
\]

**Theorem 2.7** ([A13, Theorem 4.1.2]). Let \( Y \subset X \) be \( F \)-analytic manifolds and let \( y \in Y \). Let \( \xi \in S^*(X) \) and suppose that \( \text{Supp}(\xi) \subset Y \). Then \( WF_y(\xi) \) is invariant with respect to shifts by the conormal space \( CN^X_{y,d} \).

**Corollary 2.8.** Let \( M \) be an \( F \)-analytic manifold and \( N \subset M \) be a closed algebraic submanifold. Let \( \xi \) be a distribution on \( M \) supported in \( N \). Suppose that for any \( x \in N \), we have \( CN^M_{N,x} \notin WF_x(\xi) \). Then \( \xi = 0 \).

**Proof.** Suppose \( \xi \neq 0 \) and let \( x \in \text{Supp}(\xi) \). Then \( (x, 0) \in WF_x(\xi) \). But then from Theorem 2.7 we have \( CN^M_{N,x} \subset WF_x(\xi) \) which contradicts our assumption on \( \xi \).

2.3. **Vanishing of equivariant distributions.** The following criterion for vanishing of equivariant distributions follows from [BZ76, §6] and [Ber83, §1.5].

**Theorem 2.9** (Bernstein-Gelfand-Kazhdan-Zelevinsky). Let an algebraic group \( H \) act on an algebraic variety \( X \), both defined over \( F \). Let \( \chi \) be a character of \( H \). Let \( Z \subset X \) be a closed \( H \)-invariant subset. Suppose that for any \( x \in Z \) we have
\[
\chi|_{H_x} \neq \Delta_H|_{H_x}\Delta_H^{-1},
\]
where \( \Delta_H \) and \( \Delta_{H_x} \) denote the modular functions of the groups \( H \) and \( H_x \). Then there are no non-zero \( (H,\chi) \)-equivariant distributions on \( X \) supported in \( Z \).
2.4. Characters of unipotent groups. The following lemma is standard.

**Lemma 2.10.** Let $V$ be a unipotent algebraic group defined over $F$, let $\alpha$ be a (locally constant, complex) character of $V$ and $\beta$ be a non-trivial character of $F$. Then there exists an algebraic group morphism $\varphi : V \to \mathbb{G}_a$ such that $\alpha = \beta \circ \varphi$.

For completeness we include a proof in Appendix [A]. In the case when $V$ is a maximal unipotent subgroup of a reductive group and $F$ is an arbitrary field (of an arbitrary characteristic) this lemma is [BH02 Theorem 4.1].

3. Proof of Theorem A

**Lemma 3.1.** Let $x \in G$. Let $\xi$ be a $(U, \psi)$-left equivariant and $(H, \chi)$-right equivariant distribution on $G$ such that $WF(\xi) \subset G \times \mathcal{N}_g^*$. Then $WF_x(\xi) \subset C_{N_B}^{G}$. Further, let $x \in BxH$.

**Proof.** Let $t$ be the Lie algebra of a maximal torus contained in $B$, and let $h, u$ be the Lie algebras of $H, U$ respectively. Identify $T_x^* G$ with $g^*$ using the right multiplication by $x^{-1}$. We have $C_{N_B}^{G} = (t + u + \text{ad}(x)h)^\perp$. Similarly, since $\xi$ is $u$-equivariant, by Theorem 2.6 we have $WF_x(\xi) \subset u^\perp$. Now, $u^\perp \cap \mathcal{N}_g^* = (t + u)^\perp \cap \mathcal{N}_g^* = C_{N_B}^{G}$. □

Now we would like to describe the structure of the varieties $\mathcal{O}_c$. For this we will use the following notation.

**Notation 3.2.** For a $B \times H$ double coset $\mathcal{O} = BxH \subset G$ define

$$\tilde{\mathcal{O}}_c = \bigcup_{\mathcal{O}' \subset BxH \subset \mathcal{O}} \mathcal{O}'$$

**Lemma 3.3.** For any double coset $\mathcal{O} = BxH \subset G$ there exists a closed algebraic subvariety $\tilde{\mathcal{O}}_c \subset BxH$ s.t. $\tilde{\mathcal{O}}_c = \tilde{\mathcal{O}}_c(F)$.

**Proof.** Note that

$$\mathcal{O}_c = \left\{ x \in \mathcal{O} \mid \psi x^{-1} \big|_{H \cap U x^{-1}} = \chi |_{H \cap U x^{-1}} \right\}.$$ 

Let $H_x := H \cap U x^{-1}$. Since $U$ is normal in $B$, for any $y \in (BxH)(F)$ we have $H_x = H_y$. Thus we will denote $H_{\mathcal{O}} := H_x$.

By Lemma 2.10 there exist an additive character $\beta$ of $F$ and algebraic group homomorphisms $\psi' : U \to \mathbb{G}_a$, $\chi' : H_{\mathcal{O}} \to \mathbb{G}_a$ such that $\psi = \beta \circ \psi'$ and $\chi|_{H_{\mathcal{O}}} = \beta \circ \chi'$. Let us show that

$$\tilde{\mathcal{O}}_c = \left\{ y \in (BxH)(F) \mid (\psi')y^{-1}\big|_{H_{\mathcal{O}}} = \chi' \right\}.$$ 

Indeed, if $y \in \tilde{\mathcal{O}}_c$ then $\beta \circ (\psi')y^{-1}|_{H_{\mathcal{O}}} = \beta \circ \chi'$, hence $\beta \circ (\chi' - (\psi')y^{-1}|_{H_{\mathcal{O}}}) = 1$, thus $\chi' - (\psi')y^{-1}|_{H_{\mathcal{O}}}$ is bounded on $H_{\mathcal{O}}$, and thus $\chi' - (\psi')y^{-1}|_{H_{\mathcal{O}}}$ is trivial. We obtain

$$\tilde{\mathcal{O}}_c = \left\{ y \in (BxH)(F) \mid \forall u \in H_{\mathcal{O}} \text{ we have } \psi'(uyu^{-1}) = \chi'(u) \right\},$$

which is clearly the set of $F$-points of a closed algebraic subvariety of $BxH$. □
Corollary 3.4.

(1) There exists a stratification of \( \mathcal{O}_c \) into a union of smooth \( F \)-analytic locally closed submanifolds \( \mathcal{O}_c^i \) s.t. \( \bigcup_{i \leq i_0} \mathcal{O}_c^i \) is open in \( \mathcal{O}_c \).

(2) Moreover, if \( \mathcal{O}_c \neq \mathcal{O} \) then the dimensions of \( \mathcal{O}_c^i \) are strictly smaller than the dimension of \( \mathcal{O}_c \).

Proof of Theorem A. Suppose that there exists a non-zero right \((U, \psi)\)-equivariant and left \((H, \chi)\)-equivariant distribution \( \xi \) supported on \( Z \) such that \( \text{WF}(\xi) \subset G \times N_g^* \). We decompose \( G \) into \( B \times H \)-double cosets and prove the required vanishing coset by coset. For a \( B \times H \)-double coset \( O \subset G \) define \( O_s := O \setminus \mathcal{O}_c \) and stratify \( \mathcal{O}_c \), using Corollary 3.4, to a union of smooth locally closed \( F \)-analytic subvarieties \( \mathcal{O}_c^i \). The collection

\[
\{ \mathcal{O}_c^i | O \text{ is a } B \times H \text{-double coset} \} \cup \{ O_s | O \text{ is a } B \times H \text{-double coset} \}
\]

is a stratification of \( G \). Order this collection to a sequence \( \{ S_i \}_{i=1}^{N} \) of smooth locally closed \( F \)-analytic submanifolds of \( G \) such that \( U_k := \bigcup_{i=1}^{k} S_i \) is open in \( G \) for any \( 1 \leq k \leq N \). Let \( k \) be the maximal integer such that \( \xi|_{U_{k-1}} = 0 \). Suppose \( k \leq N \) and let \( \eta := \xi|_{U_k} \). Note that \( \text{Supp}(\eta) \subset S_k \). We will now show that \( \eta = 0 \), which leads to a contradiction.

Case 1. \( S_k = O_s \) for some orbit \( O \). Then \( \eta = 0 \) by Theorem 2.9 since \( \eta \) is \((U \times H, \psi \times \chi)\)-equivariant.

Case 2. \( S_k \subset O = \mathcal{O}_c \) for some orbit \( O \). Then \( S_k \subset G \setminus Z \) and \( \eta = 0 \) by the conditions.

Case 3. \( S_k \subset \mathcal{O}_c \subset O \) for some orbit \( O \). In this case, by Corollary 3.4 \( \dim S_k < \dim O \) and thus

\[
\text{CN}_{\mathcal{O}_c}^G \supset \text{CN}_{S_k}^G.
\]

By Lemma 3.1 we have, for any \( x \in S_k \),

\[
\text{WF}_x(\eta) \subset \text{CN}_{\mathcal{O}_c}^G \text{ and thus } \text{CN}_{S_k}^G \not\subset \text{WF}_x(\eta).
\]

By Corollary 2.8 this implies \( \eta = 0 \).

\( \square \)

4. Proof of Corollaries \( \mathbb{B} \) and \( \mathbb{C} \)

Let \( U' \) denote the derived group of \( U \).

Lemma 4.1. Let \( \overline{W} \) be the Weyl group of \( G \). Let \( \overline{w} \in \overline{W} \) and let \( w \in G \) be a representative of \( \overline{w} \). Suppose that \( wUw^{-1} \cap U \subset U' \). Then \( \overline{w} \) is the longest element in \( \overline{W} \).

Proof. Let \( u \) be the Lie algebra of \( U \). On the level of Lie algebras the condition \( wUw^{-1} \cap U \subset U' \) means that \((Ad(w)u) \cap u \subset u' \). The algebra \( u \) can be decomposed as

\[
u = \bigoplus_{\alpha \in \Phi^+} g_\alpha.
\]

It is easy to see that

\[
(Ad(w)u) \cap u = \sum_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^+} g_\alpha.
\]
Let $\Delta \subset \Phi^+$ be the set of simple roots in $\Phi^+$. Then from the condition of the lemma we obtain that $w^{-1}(\Delta) \subset \Phi^-$, and as a consequence $w^{-1}(\Phi^+) \subset \Phi^-$. Let $w_0$ be the longest element in $W$. Then $w_0 w^{-1}(\Phi^+) \subset \Phi^+$. Since $\Phi^+$ is a finite set and $w_0 w^{-1}$ acts by an invertible linear transformation, we get $w_0 w^{-1}(\Phi^+) = \Phi^+$. Since simple roots are the indecomposable ones, it follows that $w_0 w^{-1}(\Delta) = \Delta$. This implies that $w_0 w^{-1} = 1$ (see e.g. [Hum72, §10.3]), and thus $w_0 = w$. □

Corollary 4.2. Let $H$ be a reductive group. Assume $G = H \times H$ and let $\Delta H \subset G$ be the diagonal copy of $H$. Denote $X = G/\Delta H$ and let $x \in X$ be such that $U_x \subset U'$. Then the orbit $Bx$ is open in $X$.

Proof. We can identify $X$ with $H$ using the projection on the first coordinate. We can assume that $B = B_H \times B_H$ where $B_H$ is a Borel subgroup of $H$. Let $W$ be the Weyl group of $H$ and $W$ be a set of its representatives. By the Bruhat decomposition,

$$H = \bigsqcup_{w \in W} B_H w B_H$$

It is well-known that the only open $B_H \times B_H$ orbit in $H$ is $B_H w_0 B_H$, where $w_0 \in W$ is the representative of the longest Weyl element. Let $w \in W$. Let $U_H$ be the nilradical of $B_H$. Then

$$U_w = \{(u_1, u_2) | u_1 w u_2 = w, \; u_1, u_2 \in U_H\},$$

and we see that for a pair $(u_1, u_2) \in U_w$ we have $u_1 = w u_2 w^{-1} \in w U_H w^{-1}$. Therefore,

$$U_w \cong U_H \cap w U_H w^{-1}.$$  

Let

$$R = \{x \in X | U_x \subset U'\} = \{x \in H | U_H \cap x U_H x^{-1} \subset U_H' = [U_H, U_H]\},$$

and let $R$ be the corresponding algebraic variety. Since $U$ and $U'$ are normal in $B$, we obtain that $R$ is $B$-invariant. The corollary follows now from Lemma 4.1. □

Corollary 4.3. Let $H \subset G$ be a subgroup of Galois type. Then for every non-open $B$-orbit $\mathcal{O} \subset G/H$ there exists $y \in \mathcal{O}$ such that $\psi(U_y) \neq 1$.

Proof. Let $\mathcal{O} \subset G/H$ be a non-open $B$-orbit and $x \in \mathcal{O}$. Then the map $B \to G$ given by the action on $x$ is not submersive and thus $B x$ is not Zariski open in $G/H$. By Corollary 4.2 this implies $U_x \not\subset U'$. Thus, there exists a non-degenerate character $\varphi$ of $U$ such that $\varphi(U_x) \neq 1$. For a fixed $x \in \mathcal{O}$, the set of characters $\varphi'$ of $U$ such that $\varphi'(U_x) \neq 1$ is Zariski-open, thus dense in the $l$-space topology and thus intersects the $B$-orbit of $\varphi$. Thus there exists $y \in B x = \mathcal{O}$ such that $\psi(U_y) \neq 1$. □

Proof of Corollary [B] By Theorem [A] it is enough to show that $S \subset Z$. Let $\mathcal{O} \subset S$ be a $B \times H$ double coset. Corollary 4.3 implies that there exists $x \in \mathcal{O}$ such that $\psi|_{W \cap H x} \neq 1$. Since $H^x$ is reductive and $U$ is unipotent, we have $\chi^x|_{U \cap H x} = 1$, and thus $\mathcal{O} \subset Z$. □

Proof of Corollary [C] Define $\tilde{G} = G \times G$, $\tilde{H} = \Delta(G) \subset \tilde{G}$ and $\tilde{B} = B \times B$. The non-degenerate characters $\psi_1, \psi_2$ define a non-degenerate character of the nilradical $\tilde{U}$ of $\tilde{B}$. Note that $\tilde{H} \subset \tilde{G}$ is a subgroup of Galois type and that $\tilde{G}/\tilde{H}$ is naturally isomorphic to $G$. Let $\eta$ be the pull-back of $\xi$ to $\tilde{G}$ under the projection $\tilde{G} \to \tilde{G}/\tilde{H} \cong G$. Then we have $\text{Supp} \; \eta \subset \tilde{S}$, where $\tilde{S}$ is the union of all non-open $\tilde{B} \times \tilde{H}$-double cosets.
in $\tilde{G}$. Also, by Corollary 2.4 we have $WF(\eta) \subset \tilde{G} \times N_{\tilde{g}}^\ast$. By Corollary B we obtain $\eta = 0$ and thus $\xi = 0$. □

**Remark 4.4.** Corollary B can not be generalized literally to arbitrary symmetric pairs. The reason is that neither can Corollary 4.3. For example consider the pair $G = GL_{2n}, H = GL_n \times GL_n$, where the involvation is conjugation by the diagonal matrix with first $n$ entries equal to $1$ and others equal to $-1$. Let $x$ be the coset of the permutation matrix given by the product of transpositions

$$\prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (2i + 1, 2n - 2i),$$

and let $B$ consist of upper-triangular matrices. Then $U_x \subset U'$, while $Bx$ is of middle dimension in $G/H$. It can be shown that there exists a $(U, \psi)$-left equivariant, $H$-right invariant distribution $\xi$ on $G$ supported in $BxH$ and satisfying $WF(\xi) \subset G \times N_{\tilde{g}}^\ast$.

However, Corollary D might hold for any spherical subgroup $H$. In fact, this is the case over the archimedean fields, see [AG, Corollary B].

**Appendix A. Proof of Lemma 2.10**

**Lemma A.1.** Let $v$ be the $(F$-points of) the Lie algebra of $V$. Then the exponential map $\exp : v \to V$ maps the commutant $[v, v]$ of $v$ onto the subgroup $[V, V]$ of $V$ generated (set-theoretically) by all the commutators in $V$.

**Proof.** Let $v_i$ be the sequence of subalgebras of $v$ defined by $v_0 := v$, $v_{i+1} := [v, v_i]$. The Baker-Campbell-Hausdorff formula implies that for any $X \in v$ and $Y \in v_i$ there exist $A, B \in v_{i+2}$ and $C \in v_{i+1}$ such that

1. $\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + A,$
2. $\log(e^X e^Y e^{-X} e^{-Y}) = [X, Y] + B$
3. $e^{X+Y} = e^C e^X e^Y.$

By (12) we have $[V, V] \subset \exp([v, v])$. To prove the opposite inclusion we prove by descending induction on $i$ that $\exp(v_i) \subset [V, V]$ for any $i > 0$. Since $\exp(v_i)$ is a group, it is enough to show that for any $X \in v$ and $Y \in v_{i-1}$ we have $\exp([X, Y]) \in [V, V]$.

Let $B$ be as in (2), and $C$ be as in (3) applied to $\log(e^X e^Y e^{-X} e^{-Y})$ and $-B$. Then $B, C \in v_{i+1}$, the induction hypothesis implies that $e^B, e^C \in [V, V]$ and thus

$$\exp([X, Y]) = \exp(\log(e^X e^Y e^{-X} e^{-Y}) - B) = e^C(e^X e^Y e^{-X} e^{-Y})e^{-B} \in [V, V].$$

□

**Corollary A.2.** Let $V/[V, V]$ denote the abelization of $V$. Then the natural map $V/[V, V] \to (V/[V, V])(F)$ is an isomorphism.

**Proof.** Let $v$ be $v$ considered as an algebraic variety. By (12), the quotient $V/\exp([v, v])$ is an abelian group. Hence $[V, V] \subset \exp([v, v])$. Thus, by Lemma A.1 we have $[V, V] \subset [V, V](F) \subset \exp([v, v]) = [V, V]$. Therefore $[V, V] = [V, V](F)$. Since unipotent groups have trivial Galois cohomologies (see [Ser97, §III.2.1, Proposition 6]), $V(F)/[V, V](F) = (V/[V, V])(F)$ and the statement follows. □
By this corollary Lemma 2.10 reduces to the case when $V$ is commutative. Since any commutative unipotent group over $F$ is a power of $G_a$, this case follows from the isomorphism of $F$ to its Pontryagin dual. \hfill $\square$

References


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