Automorphic Forms - Home Assignment 1

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Question 1

In this question we consider the Eisenstein series and its Fourier coefficients.

(a) Compute the Fourier transform

$$\int_{-\infty}^{\infty} \frac{1}{z^k} \exp(-2\pi i z\xi) \mathrm{d}z$$

of the function $\frac{1}{z^k}$.

(b) Use this to compute the Fourier expansion

$$\begin{aligned} E_k = \sum_{(m,n)=1, m\geq 0} \frac{1}{(m\tau+n)^k} &= \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k} \\ &= 1 + \frac{-2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) \exp(2\pi i n\tau), \end{aligned}$$

where k is an even integer. You may use the identity $\zeta(k) = (-1)^{k/2+1} B_k \frac{(2\pi)^k}{2\cdot k!}$ for even k, where B_k are Bernoulli's numbers. Also recall that $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

(c) Using the first few values of B_k, $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{-1}{30}$, $B_{10} = \frac{5}{66}$ and $B_{12} = \frac{-691}{2730}$, deduce the number theoretic identities:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m),$$

$$11\sigma_9(n) = -10\sigma_3(n) + 21\sigma_5(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m),$$

and show that

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

where $\Delta = \sum \tau(n) q^n$ is the modular discriminant, normalized so that $\tau(1) = 1$.

Question 2

In this question we consider Hecke operators and Euler products. Recall that the Hecke operators are defined to act on periodic functions by:

$$T_m f(z) = \sum_{ad=m} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) \frac{m^{k-1}}{d^k}.$$

(a) Prove the identities:

$$T_{mn} = T_m T_n \quad \text{for } (m,n) = 1,$$
$$T_{p^{\nu+1}} = T_p T_{p^\nu} - p^{k-1} T_{p^{\nu-1}},$$

and use them to conclude that

$$T_m T_n = \sum_{r|(m,n)} r^{k-1} T_{\frac{mn}{r^2}}.$$

(b) Let $f = \sum a_n q^n$ be a Hecke cusp form (meaning that $a_0 = 0$ and there are numbers λ_m such that $T_m f = \lambda_m f$). Assume that f is normalized so that $a_1 = 1$. Show that $a_m = \lambda_m$, and prove that if $L_f(s) = \sum a_n n^{-s}$ is the associated L-function, then

$$L_f(s) = \prod_p (1 - \lambda_p p^{-s} + p^{k-1} p^{-2s})^{-1}.$$

(c) Using the fact that Hecke operators preserve cusp forms, prove that the modular discriminant is a Hecke cusp form and show that if $|\tau(n)| \leq O(n^{\frac{11}{2}+\epsilon})$ then $|\tau(p)| \leq 2p^{\frac{11}{2}}$. Note that this implies that

$$(1 - \tau(p)p^{-s} + p^{k-1}p^{-2s}) = (1 - \alpha_p p^{\frac{k-1}{2}-s})(1 - \alpha_p^{-1} p^{\frac{k-1}{2}-s})$$

for some imaginary α_p.

(d) What about Eisenstein series? Show that for k an even integer,

$$L_{E_k}(s) = \frac{-2k}{B_k} \zeta(s)\zeta(s-(k-1)) \tag{1}$$

meaning that the L-function factorizes into two Dirichlet L-functions. Note that this L-function ignores the first coefficient a_0 of the Fourier transform of the Eisenstein series, and in particular is not exactly the Mellin transform of E_k (which diverges badly).

Question 3

In this question we consider the Fourier coefficients of the modular discriminant. Define the Eisenstein series E_2 by:

$$E_2(\tau) = 1 - \frac{4}{B_2} \sum_{n=1}^{\infty} \sigma_1(n) \exp(2\pi i n \tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) \exp(2\pi i n \tau).$$

We claim that E_2 has almost modular transformation properties:

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d), \tag{2}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. This can be proven using the properties of the associated L-function (see equation (1))

$$L_{E_2}(s) = -24\zeta(s)\zeta(s-1)$$

as follows.

(a) We first need the auxiliary identity $\Gamma(s)\Gamma(1-s)\sin(\pi s) = \pi$. You should try to prove it (but it is not very related to modular forms).

(b) Begin by showing that the completed L-function

$$M_{E_2}(s) = (2\pi)^{-s}\Gamma(s)L_{E_2}(s)$$

satisfies the functional equation:

$$M_{E_2}(2-s) = -M_{E_2}(s).$$

(You may use the functional equation $\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2})\zeta(1-s)$.)

(c) Let g be the auxiliary function

$$g(y) = E_2(iy) - 1.$$

Show that for $\Re(s) > 2$,

$$M_{E_2}(s) = \int_0^\infty g(y)y^s\frac{dy}{y}.$$

Now, use the Mellin inversion formula, which means that

$$g(y) = \frac{1}{2\pi i}\int_{2+\varepsilon-i\infty}^{2+\varepsilon+i\infty} M_{E_2}(s)y^{-s}ds,$$

(the path of integration needs to pass within the domain of absolute convergence) to show that

$$g(y) + \frac{1}{y^2}g(1/y) = \frac{1}{y^2}\sum_{-\varepsilon\leq\Re(s)\leq 2+\varepsilon} \mathrm{Res}_s(y^sM_{E_2})$$

(d) Use the Taylor expansions $\zeta(s) = \frac{1}{s-1} + \gamma + ...$ and $\Gamma(s) = \frac{1}{s} - \gamma + ...$, together with known values of Γ and ζ to show that:

$$\sum_{-\varepsilon\leq\Re(s)\leq 2+\varepsilon} \mathrm{Res}_s(y^sM_{E_2}) = -1 - y^2 + \frac{6y}{\pi},$$

and deduce equation (2) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Re(\tau) = 0$, and hence for all of $SL_2(\mathbb{Z})$ and all τ.

(e) Now, consider the modular eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. Show that:

$$\frac{1}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)} = \frac{1}{24}E_2$$

and deduce that η^{24} is a modular cusp form of weight 12. Conclude that $\Delta = \eta^{24}$. In particular, using Euler's pentagonal number theorem,

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2},$$

show that $\tau(n-1)$ is the number of ways to represent n as a sum of 24 pentagonal numbers, counted with signs.

Question 4

In this question we consider theta functions. Recall that $\theta(\tau) = \sum q^{n^2}$.

(a) Use the Poisson summation formula $\sum_{n\in\mathbb{Z}} \phi(n) = \sum_{n\in\mathbb{Z}} \hat{\phi}(n)$, where ϕ is a Schwartz function and $\hat{\phi}$ is its Fourier transform, to show that:

$$\theta(-\frac{1}{4\tau}) = \sqrt{\frac{2\tau}{i}}\theta(\tau).$$

(b) Conclude that

$$\theta\left(\frac{\tau}{-4\tau+1}\right) = (-4\tau+1)^{1/2}\theta(\tau).$$

Finally, use these results and the periodicity of θ to show that

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right)^2 = \chi(d)(c\tau+d)\theta(\tau)^2,$$

whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $c \equiv 0 \pmod 4$, and where $\chi(d)$ is the Dirichlet character

$$\chi(d) = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ -1 & \text{if } d \equiv -1 \pmod 4 \end{cases}.$$