Mathematical background for random 3SAT

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The central limit theorem, a strengthening of the law of large numbers, says that sums of independent random variables converge to the normal distribution. Here is one representative quantitative bound.

Theorem. Let $p_1, \ldots, p_n \in [0,1]$ and set $p = \frac{p_1 + \ldots + p_n}{n}$. Assume that $X_1, \ldots X_n$ are mutually independent random variables with $\Pr[X_i = 1 - p_i] = p_i$ and $\Pr[X_i = -p_i] = 1 - p_i$. Let $X = X_1 + \ldots + X_n$. (Observe that $E[X_i] = 0$, and $E[X] = 0$.) Then

$$\Pr[X > a] < e^{-a^2/2pn + a^3/2(pn)^2}$$

and

$$\Pr[X < -a] < e^{-a^2/2pn}$$

The theorem above is typically used when $p \leq 1/2$. Note that the bounds for $X < -a$ seem stronger than those for $X > a$. They can be used for $X > a$ after replacing p by $1 - p$. Namely, $\Pr[X > a] < e^{-a^2/2(1-p)n}$.

The adjacency matrix of a connected undirected graph is nonnegative, symmetric and irreducible (namely, it cannot be decomposed into two diagonal blocks and two off-diagonal blocks, one of which is all-0). As such, the Perron-Frobenius theorem implies that:

1. All its eigenvalues are real. Let us denote them by $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$.
2. Eigenvectors that correspond to different eigenvalues are orthogonal to each other.
3. The eigenvector that corresponds to λ_1 is all positive.
4. $\lambda_1 > \lambda_2$ and $\lambda_1 \geq |\lambda_n|$.

Raleigh quotients can be used in order to bound eigenvalues. Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvalues. For a nonzero vector x, let $a_i =< x, v_i >$ and hence $x = \sum a_i v_i$. Observe that:

$$\frac{x^t A x}{x^t x} = \frac{\sum \lambda_i (a_i)^2}{\sum (a_i)^2}$$

This implies that $\lambda_n \leq \frac{x^t A x}{x^t x} \leq \lambda_1$. Moreover, if x is orthogonal to v_1 then $\frac{x^t A x}{x^t x} \leq \lambda_2$.

The eigenvalues of A^2 are $(\lambda_i)^2$. The trace of A^2 implies that $\sum (\lambda_i)^2 = \sum d_i$. For d-regular graphs, this implies that the average absolute value of eigenvalues is at most $\sqrt{d}$. For random regular graphs, or random graphs that are nearly regular, it is in fact the case that with high probability over the choice of graph, all but the largest eigenvalue have absolute value $O(\sqrt{d})$. This can be proved by the trace method (considering higher even powers of A).