Handout 3: Randomized Rounding and SemiDefinite Programming

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http://www.wisdom.weizmann.ac.il/∼dinuri/courses/11-BoundaryPNP/

We shall start with randomized rounding of the primal LP for set-cover. Thereafter, we shall consider the max-cut problem. This part follows the first 42 slides in
http://www.wisdom.weizmann.ac.il/∼feige/Slides.sdpslides.ppt.

We shall illustrate the difficulty of using LP-relaxations to obtain good approximations for max cut. We shall show a method of strengthening the relaxations by additional constraints, and comment on its effectiveness in special families of graphs. Thereafter we shall consider heuristics for max cut based on spectral techniques. This will lead us (through a duality notion for semidefinite programs) to the SDP of Goemans and Williamson (GW) and the random hyperplane rounding technique. This last part is discussed in detail in Chapter 6 (Sections 6.1 and 6.2) of the book by Williamson and Shmoys.

Homework.

1. Design an algorithm that finds maximum weight independent sets in bipartite graphs. You may use as a blackbox the existence of polynomial time algorithms for linear programming. To do so, show that for every bipartite graph, the LP for vertex cover (the primal LP discussed last week) has an optimal solution that is integral (every variable has value either 0 or 1), and moreover, that such a solution can be found in polynomial time. [Hint: show that given any feasible solution in which some variables are not integral, one can lower the values of some of the variables and raise the values of others in a way that maintains feasibility and does not increase the value of the solution. Naturally, you will need to use the fact that the graph is bipartite.]

2. Suppose that a graph with m edges is "nearly bipartite". Namely, for some tiny $\epsilon > 0$, it has a cut with $(1-\epsilon)m$ edges. Show that in this case the random hyperplane rounding technique when applied to the GW SDP finds a cut of expected size at least $(1-\delta)m$, where $\delta = O(\sqrt{\epsilon})$. Moreover, show that this quadratic relation between δ and ϵ is essentially best possible. Namely, show that for arbitrarily small $\delta > 0$ there are graphs whose true maximum cut has $(1-\delta)m$ edges, but the value of the GW SDP is $(1-\epsilon)m$ for $\epsilon = O(\delta^2)$. [Hint – consider graphs which are odd cycles of length $\Omega(1/\delta)$.]

Open questions. Recall that for VC we have seen rounding techniques for the primal LP and for the dual LP, and both give an approximation ratio of 1/2. Moreover, we have also seen a simple primal-dual algorithm (that did not require finding an optimal solution to an LP) with an approximation ratio of 1/2. For max cut and using the SDP approach, we have seen a rounding technique of the "primal" GW SDP that gives an approximation ratio of roughly 0.878. Is there also a natural way to round the dual SDP (the one based on adding the minimum number of self-loops so as to make the adjacency matrix positive semidefinite) giving a similar approximation ratio? Is there a primal-dual algorithm with a similar approximation ratio that does not require solving an SDP? Partial progress on such questions was achieved in [Luca Trevisan: Max cut and the smallest eigenvalue. STOC 2009: 263–272].