In this exercise we study some analysis of Boolean functions. For references one can find several online courses on this subject.

Questions:

1. Given a string $x \in \{-1, 1\}^n$, let $N_\varepsilon(x)$ be the distribution on strings in $\{-1, 1\}^n$ obtained by flipping each coordinate of $x$ independently with probability $\varepsilon$. Define an operator $T$ taking a function $f : \{-1, 1\}^n \to \{-1, 1\}$ to a new function $g = Tf : \{-1, 1\}^n \to \{-1, 1\}$ by
   \[
   \forall x \in \{-1, 1\}^n, \quad g(x) = E_{y \sim N_\varepsilon(x)} f(y).
   \]
   (a) Prove that for every function $f$, $E_x[f(x)] = E_x[Tf(x)]$.
   (b) Prove that if $f$ is Boolean then for each $x$, $|Tf(x)| \leq 1$.
   (c) Express the Fourier coefficients of $g = Tf$ as a function of those of $f$.

2. We define the influence of the $i$'th variable on a function $f : \{0, 1\}^n \to \{0, 1\}$ to be
   \[
   \text{Inf}_i(f) := P_{x \in \{0, 1\}^n} [f(x) \neq f(x \oplus e_i)].
   \]
   The total influence of a function is the sum of all influences, $I(f) = \sum_{i=1}^n \text{Inf}_i(f)$. Compute the influences and the total influence of the following functions:
   (a) dictator: $f(x_1, \ldots, x_n) = x_1$.
   (b) junta: $f(x_1, \ldots, x_n) = x_1 \text{ AND } x_2$.
   (c) parity: $f(x_1, \ldots, x_n) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$
   (d) majority: $f(x_1, \ldots, x_n) = 1$ iff $x_1 + x_2 + \cdots + x_n > n/2$ (an approximate computation is ok).

3. Prove the following formula for $I(f)$:
   \[
   I(f) = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2
   \]
   (hint: consider the Fourier expansion of the function $f_i(x) = f(x) \oplus f(x \oplus e_i)$.)