On the Boundary between P and NP Spring 2011

Problem Set # 5

Due: May 2nd, 2011 *Lecturer: Irit Dinur*

In this exercise we study some analysis of Boolean functions. For references one can find several online courses on this subject.
Questions:

1. Given a string $x \in \{-1,1\}^n$, let $N_\varepsilon(x)$ be the distribution on strings in $\{-1,1\}^n$ obtained by flipping each coordinate of x independently with probability ε. Define an operator T taking a function $f : \{-1,1\}^n \to \{-1,1\}$ to a new function $g = Tf : \{-1,1\}^n \to \{-1,1\}$ by
$$\forall x \in \{-1,1\}^n, \quad g(x) = E_{y \sim NS_\varepsilon(x)} f(y).$$
   (a) Prove that for every function f, $E_x[f(x)] = E_x[Tf(x)]$.
   (b) Prove that if f is Boolean then for each x, $|Tf(x)| \leq 1$.
   (c) Express the Fourier coefficients of $g = Tf$ as a function of those of f.

2. We define the influence of the i'th variable on a function $f : \{0,1\}^n \to \{0,1\}$ to be
$$Inf_i(f) := Pr_{x \in \{0,1\}^n}[f(x) \neq f(x \oplus e_i)].$$
The total influence of a function is the sum of all influences, $I(f) = \sum_{i=1}^n I_i(f)$. Compute the influences and the total influence of the following functions:
   (a) dictator: $f(x_1, \dots, x_n) = x_1$.
   (b) junta: $f(x_1, \dots, x_n) = x_1 \; AND \; x_2$.
   (c) parity: $f(x_1, \dots, x_n) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$
   (d) majority: $f(x_1, \dots, x_n) = 1$ iff $x_1 + x_2 \cdots + x_n > n/2$ (an approximate computation is ok).

3. Prove the following formula for $I(f)$:
$$I(f) = \sum_{S \subseteq [n]} |S| \, \hat{f}(S)^2$$
(hint: consider the Fourier expansion of the function $f_i(x) = f(x) \oplus f(x \oplus e_i)$.)