You may consult relevant references, but should write your answers on your own (no copy-paste from other sources). It is advisable to include drawings in your answers, wherever applicable. (No need to spend much time on producing electronic drawings - they can be drawn by hand.) Please hand in your exam on or before July 22, 2011.

1. The **sparsity** of a cut induced by a set $S$ of vertices in an unweighted graph $G = (V, E)$ is the ratio between the number of edges $|E(S, V \setminus S)|$ crossing the cut and the product $|S| \cdot |V \setminus S|$ of sizes. Show that for some $\epsilon > 0$ it is $\text{R3SAT}$-hard (in the sense of [3]) to approximate **sparsest cut** within a ratio better than $1 + \epsilon$. In designing the reduction and proving its correctness, you may consult [1] (or versions of that paper available on the home pages of its authors).

2. For a $d$-regular unweighted graph $G = (V, E)$, let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its normalized eigenvalues, i.e. the eigenvalues of the matrix whose $i,j$-th entry is $1/d$ if there is an edge from $i$ to $j$ and zero otherwise.

Prove that for some constant $c > 0$ and every $\epsilon, \beta > 0$ there is a polynomial-time algorithm that when given as input a $d$-regular graph $G$ over $n$ vertices with at least $n^{c\beta}$ normalized eigenvalues above $1 - \epsilon$ it outputs a set $S$ satisfying

(a) $S$ is small: The size of $S$ is at most $n/n^\beta$.
(b) $S$ is non-expanding: The expansion of $S$ is at most

$$\frac{|E(S, V \setminus S)|}{d|S|} \leq \sqrt{\epsilon/\beta}.$$ 

You may consult [2].

**References**

