PCPs and HDX - Homework 1

Due: December 6, 2016

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTex.

1 Transformations on constraint systems

Recall that a system of constraints is a triple $(V, \mathbf{C}, \Sigma)$ where V is a set of vertices, Σ is an alphabet, and $\mathbf{C}$ is a set of constraints. Each constraint in $\mathbf{C}$ is a pair (τ, φ) where $\tau = (v_1, \ldots, v_q)$ is a tuple of elements in V and $\varphi : \Sigma^q \to \{0, 1\}$ is a Boolean predicate.

A function $f : V \to \Sigma$ satisfies a constraint (τ, φ) if

$$\varphi(f(v_1), \ldots, f(v_q)) = 1$$

Warmup: Write the system of constraints corresponding to the 3SAT formula whose clauses are

$$(x_1 \vee \neg x_2 \vee \neg x_3), (x_2 \vee \neg x_3 \vee x_4), (x_1 \vee x_3 \vee x_4)$$

What is Σ ? What is the arity of the constraints (the arity is the number of variables in each constraint)?

Notation: Recall that $\mathrm{SAT}(\mathbf{C}) = \{f : V \to \Sigma \mid f \text{ satisfies every cosntraint in } \mathbf{C}\}$ and $\mathrm{rej}_{\mathbf{C}}(f) = \Pr_{c \in \mathbf{C}}[f \text{ satisfies } c]$. A constraint system is γ-expanding if for every $f : V \to \Sigma$, $\mathrm{rej}_{\mathbf{C}}(f) \geq \gamma \cdot \mathrm{dist}(f, \mathrm{SAT}(\mathbf{C}))$.

2 The agreement constraint system

Let $(V, \mathbf{C}, \Sigma)$ be a system of q-ary constraints. Assume that each variable appears in at least one constraint. We describe a transformation from $\mathbf{C}$ to a new system $(\widehat{V}, \widehat{\mathbf{C}}, \widehat{\Sigma})$ such that $\widehat{V} = \mathbf{C}$ and each new variable takes values in $\widehat{\Sigma} := \Sigma^q$. The constraints in $\widehat{\mathbf{C}}$ are 2-ary and described as follows.

Let us refer to V as the old variables and to $\mathbf{C}$ as the old constraints or the new variables depending on the context. An value $a \in \widehat{\Sigma}$ assigned to a new variable c is interpreted as a tuple $a = (a_1, \ldots, a_q)$ of old values assigned to the old variables that participate in c. For each old variable $v \in V$ and every pair of old constraints $c_1, c_2 \in \mathbf{C}$ that touch this variable, we will have a new constraint in $\widehat{\mathbf{C}}$. This constraint will look at the two (new) variables corresponding to c_1 and c_2, and do the following

- Check that the assignment to c_1, which is some $(a_1, \ldots, a_q) \in \widehat{\Sigma} = \Sigma^q$ satisfies the old constraint c_1. Similarly for c_2.

- Check agreement: verify that the common variable v that occurs in both c_1 and c_2 is assigned the same value, i.e. that the two assignments *agree* on the common variable.

If both checks succeed the constraint will be satisfied, otherwise it will not.

1. Prove that there is a bijection from $\mathrm{SAT}(\mathbf{C})$ to $\mathrm{SAT}(\widehat{\mathbf{C}})$ by describing a mapping $Enc : \Sigma^V \to \widehat{\Sigma}^{\widehat{V}}$ that maps assignments for V to assignments for $\widehat{V}$.

2. Suppose that $\mathbf{C}$ is regular, i.e. every variable occurs in the same number D of constraints. How many constraints does $\widehat{\mathbf{C}}$ have? Is $\widehat{\mathbf{C}}$ regular?

   Given any $\widehat{f} : \widehat{V} \to \widehat{\Sigma}$, that is not necessarily in $\mathrm{SAT}(\widehat{\mathbf{C}})$, define the majority decoding assignment $f : V \to \Sigma$. ~~Show that $\mathrm{dist}(\widehat{f}, \mathrm{SAT}(\widehat{\mathbf{C}})) \geq \mathrm{dist}(\widehat{f}, Enc(f))$.~~ Show a counter example to the (striked out) claim, i.e. show an example constraint system where $\mathrm{dist}(\widehat{f}, \mathrm{SAT}(\widehat{\mathbf{C}})) < \mathrm{dist}(\widehat{f}, Enc(f))$.

3. Suppose for the rest of this exercise that $\mathbf{C}$ is regular, and that its expansion is some $\gamma > 0$, i.e. there is some $\gamma > 0$ such that for every $f : V \to \Sigma$,

$$\mathrm{rej}_{\mathbf{C}}(f) \geq \gamma \cdot \mathrm{dist}(f, \mathrm{SAT}(\mathbf{C})).$$

   Prove that $\widehat{\mathbf{C}}$ is expanding. The following steps might be helpful. Fix $\widehat{f} : \widehat{V} \to \widehat{\Sigma}$. Denote $\rho = \mathrm{rej}_{\widehat{C}}(\widehat{f})$.

   (a) Let δ_1 be the fraction of vertices $c \in \widehat{V}$ for which $\widehat{f}(c)$ is a value that fails to satisfy c. Here we view c also as a *constraint* in the original system $\mathbf{C}$. Prove that $\delta_1 \leq \rho$.

   (b) Let f be the majority decoding of $\widehat{f}$ and let δ_2 be the distance of $\widehat{f}$ from $Enc(f)$. Prove that $\delta_2 \leq 2q\rho$.

   (c) Prove that $\mathrm{rej}_{\mathbf{C}}(f) \leq \delta_1 + \delta_2$.

   (d) Prove that $\mathrm{dist}(\widehat{f}, \mathrm{SAT}(\widehat{C})) \leq \mathrm{dist}(\widehat{f}, Enc(f)) + \mathrm{dist}(f, \mathrm{SAT}(C))$.

   (e) Use the expansion of $\mathbf{C}$ and collect the above items to prove that $\mathrm{rej}_{\widehat{\mathbf{C}}}(\widehat{f}) \geq \beta \cdot \mathrm{dist}(\widehat{f}, \mathrm{SAT}(\widehat{\mathbf{C}}))$ for some constant β that depends only on q and on γ.

3 Degree reduction

Let $(V, \mathbf{C}, \Sigma)$ be a system of 2-ary (aka binary) constraints.

1. Describe $\mathbf{C}$ as a directed graph that has a predicate associated with each edge. A predicate $\varphi : \Sigma^2 \to \{0,1\}$ is symmetric if $\varphi(x,y) = \varphi(y,x)$ for all $x, y \in \Sigma$. Show that if the predicates are symmetric then the graph need not be directed.

   We call this graph *the constraint graph.*

   In the rest of this exercise let us assume that the constraint graph is undirected (it changes nothing but might make it simpler to think about).

2. In any graph, denote $deg(v)$ to be the degree of the vertex v, which is the number of edges (incoming or outgoing) touching v. Suppose that the constraint graph is D-regular, i.e. for every v, $deg(v) = D$.

   Show a transformation from $(V, \mathbf{C}, \Sigma)$ to $(V', \mathbf{C}', \Sigma)$ such that the degree of every vertex in V' is 3 and such that

   - (Completeness:) There is a bijection from $\mathrm{SAT}(\mathbf{C})$ to $\mathrm{SAT}(\mathbf{C}')$.

- (Soundness:) Assuming that $\gamma(\mathbf{C}) \geq \gamma_0$, prove a lower bound on the expansion of $\mathbf{C}'$ that does not depend on $n = |V|$.

  Hint: you can find the following useful. I am assuming that V' is obtained from V by duplicating each variable a number of times, and that the edges of C' have two kinds: inner and across. Define the majority decoding f. Relate the rejection of f to two kinds of across edges in C': edges that reject f' and edges that don't. The first type are bounded by the rejecting across edges, and the second type are bounded by the rejecting inner edges.