PCPs and HDX - Homework 2

Due: January 3, 2017

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTex. Please email your solution to `inbal.livni@weizmann.ac.il`.

1 Graphs and eigenvalues

Let G be a d-regular graph. The Normalized Adjacency Matrix of a graph G, denoted A, is an $n \times n$ matrix that for each edge uv contains the number of edges in G between vertex u and vertex v, divided by d. Since the graph is d-regular, the sum of each row and column in A is 1. By definition the matrix A is symmetric and therefore has an orthonormal basis of eigenvectors $v_0, \dots, v_{n-1}$ with eigenvalues $\lambda_0, \dots, \lambda_{n-1}$ such that for all i we have $Av_i = \lambda_i v_i$. Without loss of generality we assume the eigenvalues are sorted in descending order $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$.

The eigenvalues of A are called the spectrum of the graph G. The spectrum of a graph contains a lot of information regarding the graph. The next exercise confirms some examples of observations that demonstrate this connection between the spectrum of a d-regular graph and its properties. Prove the following statements:

1. $\lambda_0 = 1$
2. The graph is connected iff $\lambda_0 > \lambda_1$
3. The graph is bipartite iff $|\lambda_0| = |\lambda_{n-1}|$

The graph's second largest eigenvalue is defined to be $\lambda(G) = \max(|\lambda_1|, |\lambda_{n-1}|)$ and is related to the expansion parameter of the graph.

4. Let G be the complete graph on n vertices. Find the eigenvalues of A. What is $\lambda(A)$?

2 Expander graphs and mixing

A graph is called a λ-expander if $\lambda(G) \leq \lambda$.

2.1 Spectral gap and mixing: the expander mixing lemma

We will prove that for every $S, T \subseteq V$:

$$|E(S,T) - \frac{d|S||T|}{n}| \leq \lambda d \sqrt{|S||T|}$$

where $E(S,T)$ is the number of edges between S and T and where $\lambda = \max(|\lambda_1|, |\lambda_{n-1}|)$. This is called the expander mixing lemma, and is important because it shows that the behavior of an expander graph is like a random graph: the number of edges you expect to cross from S to T if the graph were random is roughly the number of edges that you see, up to an error that is controlled by the second eigenvalue. Prove the following steps,

1. Let $\mathbf{1}_S, \mathbf{1}_T$ be the characteristic vectors of the subsets S and T (i.e. $\mathbf{1}_S \in \mathbb{R}^V$ is a vector with ones for all $v \in S$ and zeros elsewhere). Show that $\frac{1}{d} \cdot E(S,T) = \mathbf{1}_S A \mathbf{1}_T$.

2. Express $\mathbf{1}_S = \sum_i \alpha_i v_i, \mathbf{1}_T = \sum_i \beta_i v_i$ in terms of the basis of eigenvectors $v_0, \ldots, v_{n-1}$. Compute α_0, β_0 in terms of $|S|, |T|$.

3. Expand $\mathbf{1}_S A \mathbf{1}_T$ in this basis. Deduce that

$$|\frac{1}{d} \cdot E(S,T) - \frac{|S||T|}{n}| \leq |\sum_{i=1}^{n-1} \lambda_i \alpha_i \beta_i|$$

4. Use Cauchy Schwartz inequality $\sum_i \alpha_i \beta_i \leq \sqrt{\sum_i (\alpha_i)^2 \cdot \sum_i (\beta_i)^2} = \|\mathbf{1}_S\| \cdot \|\mathbf{1}_T\|$ to prove the lemma.

2.2 Spectral gap and edge expansion

In this exercise we prove the "easy direction" of the Alon-Milman-Cheeger inequality that relates the edge expansion of a graph to its spectral gap. This is the important direction for us, because it shows that if a graph has spectral gap at least γ, then it's edge expansion is at least γ.

The edge expansion of a graph is usually defined as

$$\phi(G) = \inf_{S \neq \phi, V} \frac{E(S, V \setminus S)}{\frac{d}{n} \cdot |S| \cdot |V \setminus S|}$$

Recall from the first lecture that we defined

$$\gamma(G) = \inf_{f: V \to \{0,1\},\ f \neq \{\mathbf{0},\mathbf{1}\}} \frac{\text{rej}_G(f)}{\text{dist}(f, \{\mathbf{0}, \mathbf{1}\})}$$

Prove that

$$\gamma(G)/2 \leq \phi(G) \leq \gamma(G).$$

Let $f : V \to \mathbb{R}$.

1. Prove that if $f = \mathbf{1}_S$ for some set $S \subset V$ then

$$\langle f, (I - A) f \rangle = \frac{1}{d} \cdot E(S, V \setminus S)$$

where $\langle f, g \rangle = \sum_{v \in V} f(v) g(v)$.

2. Prove, by decomposing f according to the eigenvectors of A that

$$\sup_{f \in X_0} \frac{\langle f, Af \rangle}{\langle f, f \rangle} = \lambda_1$$

where $X_0 = \{f \neq 0 \mid f \perp 1\}$

3. Prove that $1 - \lambda_1 \leq \phi(G)$. This is called the "easy direction" of Cheeger's inequality. It is also the useful direction, because it shows how the eigenvalue gap can be used to lower bound the edge expansion of a graph.