PCPs and HDX - Homework 3

Due: January 17, 2017

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTeX. Please email your solution to inbal.livni@weizmann.ac.il.

1 The Sparse Random Complex

Let $X(0)$ be a set of $n$ vertices, and let $d$ be a parameter. A triangle is a set of three distinct vertices in $X(0)$. For each $v \in X(0)$ let us choose $d$ triangles containing $v$, uniformly and independently. Let $X(2)$ be the collection of at most $nd$ triangles chosen in this way. Let $X(1)$ be all edges that participate in a triangle of $X(2)$.

1. Show that the probability of repetition, i.e. of two vertices choosing the same triangle, is small. Show that with high probability we have $nd$ triangles.

2. Show that if $d \ll n$ then with high probability the link of a vertex $v$ is highly disconnected. Describe how a typical link looks like. (conclude that this is not a high-dimensional expander).

2 The Paths Complex

Let $G = (V, E)$ be a $d$-regular expander. We construct a 2 dimensional complex by setting $X(0) = V$, and constructing $X(2)$ by placing a triangle on every triple of distinct vertices $u, v, w$ such that $uv, vw \in E$ (i.e. where $uvw$ is a length-2 path). We let $X(1)$ be the edges of the triangles described above.

1. Show that $X(1) \supset E$. Who are the edges in $X(1) - E$ ?

2. Show that $|X(2)| = O(nd^2)$.

3. Prove that the link of each vertex $v \in X(0)$ is a connected graph.

4. The girth of a graph is the length of its shortest cycle. Assume that the girth of $G$ is at least 5. Fix $v \in X(0)$ and let $X_v$ be its link.
   - Give a full description of the link of a vertex $v$.
   - Find a set $S$ of about half of the vertices in $X_v$ that have only $O(|S|)$ outgoing edges (and not $\theta(d|S|)$). Deduce an upper bound on the edge expansion of $X_v$.

Conclusion: This complex has connected links, which is better than the random complex. However, the expansion inside a link is weak in that it is not a constant independent of the number of vertices in the link. Such a graph is not considered an expander because in an expander the expansion is supposed to be independent of the number of vertices in the graph.
3 The Grassmann Graph

Fix a finite field $F = \mathbb{F}_q$, and let $V$ be the collection of all 2-dimensional linear subspaces of $\mathbb{F}^m$. This is sometimes denoted $Gr(\mathbb{F}^m, 2)$. Connect $u, v$ by an edge if $\dim(u \cap v) = 1$. Let $G$ be the resulting graph. Our goal in this exercise is to calculate $\lambda(G)$.

If it helps the calculations you can do them asymptotically for $q \rightarrow \infty$.

1. (preliminary:) Let $A$ be a matrix with eigenvectors $v_0, \ldots, v_{n-1}$ and corresponding eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$. What are the eigenvalues and eigenvectors of the matrix $B = A^2$?

2. Calculate the degree of a vertex $v \in V$.

3. Let $M$ be the normalized adjacency matrix of $G$. Write

$$M^2 = p_0 J + p_1 M + (1 - p_0 - p_1) I$$

where $I$ is the identity matrix, and $J$ is the matrix with all ones all of whose entries are $1/n$. Calculate the values for $p_0, p_1$.

4. We know the value for $\lambda(I), \lambda(J)$. We also know that $\lambda(M^2) = \lambda(M)^2$. Use these to calculate the value for $\lambda(M)$.

5. (addendum (bonus):) Let $G' = (P, V, E)$ be the bipartite graph defined by $p = \mathbb{F}^m - \{0\}$, $V = Gr(\mathbb{F}^m, 2)$ as before, and we put an edge between $x \in P$ and $v \in V$ if $x \in v$. Let $B$ be the $P \times V$ matrix such that $B(x, v) = 1$ if $x \in v$ and 0 otherwise. Find an algebraic relation between $B$ and the adjacency matrix of $G$ and of $G'$.

4 (bonus) The Spherical Building

Fix a finite field $F = \mathbb{F}_q$, and let $X(0)$ be the collection of all non-trivial linear subspaces of $\mathbb{F}^4$. Here non-trivial means of dimension $1, 2, 3$. We color a subspace $v \in X(0)$ with the color $i \in \{1, 2, 3\}$ if its dimension as a linear subspace is $i$.

We define the “edges” $X(1)$ to be the collection of pairs $u, v$ such that $u \subset v$.

We define the “triangles” $X(2)$ to be the collection of triples $u, v, w$ such that $u \subset v \subset w$.

1. Calculate the degree of a vertex $v \in X(0)$ for each of the three colors.

2. Find two large sets $S, T \subset X(0)$ such that there are no edges between $S$ and $T$. (This seems to indicate that the 1-skeleton of $X$ is not a good expander, however, we will see that not all is lost).

3. What is $\lambda(G)$ where $G$ is the 1-skeleton of $X$?

4. Describe the link of a vertex $v$ colored 1. Similarly for colors 2, 3.

5. Show that for every set $S \subset X(0)$, $|S| \leq \alpha|X(0)|$, the fraction of edges touching $S$ that stay inside $S$ is small, i.e. proportional to $\alpha$. Assume that $q \rightarrow \infty$ whereas $\alpha$ is fixed.

Hint: Split $S$ into colors. Try to express the number of edges inside $S$ using linear algebra in terms of an appropriate adjacency matrix.