PCPs and HDX - Homework 3

Due: January 17, 2017

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTex. Please email your solution to `inbal.livni@weizmann.ac.il`.

1 The Sparse Random Complex

Let $X(0)$ be a set of n vertices, and let d be a parameter. A triangle is a set of three distinct vertices in $X(0)$. For each $v \in X(0)$ let us choose d triangles containing v, uniformly and independently. Let $X(2)$ be the collection of at most nd triangles chosen in this way. Let $X(1)$ be all edges that participate in a triangle of $X(2)$.

1. Show that the probability of repetition, i.e. of two vertices choosing the same triangle, is small. Show that with high probability we have nd triangles.
2. Show that if $d \ll n$ then with high probability the link of a vertex v is highly disconnected. Describe how a typical link looks like. (conclude that this is not a high-dimensional expander).

2 The Paths Complex

Let $G = (V, E)$ be a d-regular expander. We construct a 2 dimensional complex by setting $X(0) = V$, and constructing $X(2)$ by placing a triangle on every triple of distinct vertices u, v, w such that $uv, vw \in E$ (i.e. where uvw is a length-2 path). We let $X(1)$ be the edges of the triangles described above.

1. Show that $X(1) \supset E$. Who are the edges in $X(1) - E$?
2. Show that $|X(2)| = O(nd^2)$.
3. Prove that the link of each vertex $v \in X(0)$ is a connected graph.
4. The *girth* of a graph is the length of its shortest cycle. Assume that the girth of G is at least 5. Fix $v \in X(0$ and let X_v be its link.
   - Give a full description of the link of a vertex v.
   - Find a set S of about half of the vertices in X_v that have only $O(|S|)$ outgoing edges (and not $\theta(d|S|)$). Deduce an upper bound on the edge expansion of X_v.

Conclusion: This complex has connected links, which is better than the random complex. However, the expansion inside a link is weak in that it is not a constant *independent* of the number of vertices in the link. Such a graph is not considered an expander because in an expander the expansion is supposed to be independent of the number of vertices in the graph.

3 The Grassmann Graph

Fix a finite field $\mathbb{F} = \mathbb{F}_q$, and let V be the collection of all 2-dimensional linear subspaces of $\mathbb{F}^m$. This is sometimes denoted $Gr(\mathbb{F}^m, 2)$. Connect u, v by an edge iff $dim(u \cap v) = 1$. Let G be the resulting graph. Our goal in this exercise is to calculate $\lambda(G)$.

If it helps the calculations you can do them asymptotically for $q \to \infty$.

1. (preliminary:) Let A be a matrix with eigenvectors $v_0, \ldots, v_{n-1}$ and corresponding eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$. What are the eigenvalues and eigenvectors of the matrix $B = A^2$?

2. Calculate the degree of a vertex $v \in V$.

3. Let M be the normalized adjacency matrix of G. Write

$$M^2 = p_0 J + p_1 M + (1 - p_0 - p_1)I$$

   where I is the identity matrix, and J is the matrix ~~with all ones~~ all of whose entries are $1/n$. Calculate the values for p_0, p_1.

4. We know the value for $\lambda(I), \lambda(J)$. We also know that $\lambda(M^2) = \lambda(M)^2$. Use these to calculate the value for $\lambda(M)$.

5. (addendum (bonus):) Let $G' = (P, V, E)$ be the bipartite graph defined by $p = \mathbb{F}^m - \{0\}$, $V = Gr(\mathbb{F}^m, 2)$ as before, and we put an edge between $x \in P$ and $v \in V$ if $x \in v$. Let B be the $P \times V$ matrix such that $B(x, v) = 1$ iff $x \in v$ and 0 otherwise. Find an algebraic relation between B and the adjacency matrix of G and of G'.

4 (bonus) The Spherical Building

Fix a finite field $\mathbb{F} = \mathbb{F}_q$, and let $X(0)$ be the collection of all non-trivial linear subspaces of $\mathbb{F}^4$. Here non-trivial means of dimension $1, 2, 3$. We color a subspace $v \in X(0)$ with the color $i \in \{1, 2, 3\}$ if its dimension as a linear subspace is i.

We define the "edges" $X(1)$ to be the collection of pairs u, v such that $u \subset v$.

We define the "triangles" $X(2)$ to be the collection of triples u, v, w such that $u \subset v \subset w$.

1. Calculate the degree of a vertex $v \in X(0)$ for each of the three colors.

2. Find two large sets $S, T \subset X(0)$ such that there are no edges between S and T. (This seems to indicate that the 1-skeleton of X is not a good expander, however, we will see that not all is lost).

3. What is $\lambda(G)$ where G is the 1-skeleton of X ?

4. Describe the link of a vertex v colored 1. Similarly for colors $2, 3$.

5. Show that for every set $S \subset X(0)$, $|S| \leq \alpha |X(0)|$, the fraction of edges touching S that stay inside S is small, i.e. proportional to α. Assume that $q \to \infty$ whereas α is fixed.

   Hint: Split S into colors. Try to express the number of edges inside S using linear algebra in terms of an appropriate adjacency matrix.