PCPs and HDX - Lecture 7

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1 Edge Expansion of Graphs

Let $G = (V, E)$. Let $(S, V \setminus S)$ be a cut. We are interested in the number edges in this cut. More precisely, we are interested in bounding the number of edges in this cut as a function of $\alpha = |S|/|V|$. Clearly, if the graph has no multiple edges, there are at most $|S| \cdot |V \setminus S| = \alpha(1-\alpha)n^2$ edges in the cut. This is indeed the number of edges in the complete graph. As a fraction of the number of edges, it is $2\alpha(1-\alpha)$.

For a fixed G, and a set S of size $\alpha|V|$ we are interested in the size of a typical $E(S, V \setminus S)$. We think of choosing S at random by placing each vertex of V in S with probability α independently (this simulates going over all $(\alpha, 1-\alpha)$ cuts, but easier to analyze). For an edge uv we have the following:

$$\Pr_S[uv \text{ crosses cut}] = 2\alpha(1-\alpha).$$

This implies the following,

$$\mathbb{E}_S[|E(S, V \setminus S)|] = \sum_{uv \in E} \Pr_S[uv \text{ crosses cut}] = \frac{nd}{2} \cdot 2\alpha(1-\alpha) = nd\alpha(1-\alpha),$$

or in other words, $2\alpha(1-\alpha)$ fraction of the edges. Note that this just like in the case of the complete graph.

Definition 1.1. *G is $\varepsilon-$typical with respect to S if,*

$$|E(S, V \setminus S) - \alpha(1-\alpha)nd| \leq \varepsilon \cdot nd\alpha(1-\alpha).$$

In other words, if we denote $\frac{|E(S,V \setminus S)|}{nd|S||V \setminus S|}$ by $\Phi(S)$ then, we have $|\Phi(S) - 1| \leq \varepsilon$, i.e., there is a typical fraction of edges crossing this cut.

Any graph is typical with respect to most cuts (follows from a concentration bound, showing that for most graphs the number of edges crossing between S and $V \setminus S$ is near its expectation), but not necessarily all. Is there a graph that is typical with respect to all cuts? The answer is yes: the complete graph. By counting the number of edges crossing a cut we get very exact information about the size of the two sides of the cut. Next we ask: is there such a graph that is sparse, i.e. with $O(n)$ edges and not $\Omega(n^2)$? Such a graph would behave *like* the complete graph, without needing so many edges. Some candidates are:

1. A random graph.

2. An expander.

2 Random Graphs

What is a random graph on n vertices? One model is to choose a graph out of all possible graphs over n vertices. A much more useful way to describe this is the $G(n, \frac{1}{2})$ model, or more generally, the Erdos-Renyi $G(n,p)$ model, where $0 < p < 1$ is a parameter. In this model, a graph is chosen by including each edge independently with probability p. In this model the average degree is pn.

A slightly different model is the $G(n,d)$ model, in which we pick d matchings on n vertices at random, and include all of them in the edge set. In this model the degree (counting multiplicity) of each vertex is exactly d.

For $d = pn$ these models are similar, but not identical. The $G(n,p)$ model is easier to analyze because of the independence of the edges. Let us fix d and set $p = d/n$ and consider an increasing sequence of n's. In $G \sim G(n,p)$ we are likely to have isolated vertices, but not so in $G \sim G(n,d)$. Indeed, the following is true:

Lemma 2.1 (A random graph $G \sim G(n,d)$ is an expander)**.** *For all ε, there exists a constant d such that for all large enough n and G sampled from $G(n,d)$ we have that* <u>*all*</u> *the cuts in G are $\varepsilon-$typical.*

We will prove an easier claim, pertaining to $G(n,p)$:

Lemma 2.2 (A random graph in $G \sim G(n, p = d/n)$ is a "large-set-expander")**.** *For all ε, there exists a constant d such that for all large enough n and G sampled from $G(n,p)$, where $p = d/n$ we have that all the* <u>*large*</u> *cuts in G are $\varepsilon-$typical (by large we mean that $|S| \in \left(\frac{n}{4}, \frac{3n}{4}\right)$).*

Proof. We will use an union bound argument. For all large S let $A_S =$"S is atypical". We denote $P_S = \Pr(A_S) = \exp(-\varepsilon mp(1-p))$, where $m = |S| \cdot |V \setminus S|$. We note that,

$$\Pr\left[\left|\sum x_i - mp\right| > \varepsilon mp\right] \leq 2 \cdot \exp\left(-\frac{\varepsilon^2 mp}{3}\right),$$

where $x_i = x_{uv} =$ did uv get into G. Next, we have that $\mathbb{E}[x_i] = p = d/n$. Since $m = \#uv = \alpha(1-\alpha)n^2$, we have that $mp = \alpha(1-\alpha)nd$. Thus if d is large enough with respect to ε we have:

$$\frac{\varepsilon^2 mp}{3} > \frac{nd}{16}\varepsilon^2 > n.$$

So, we have:

$$\Pr\left[\exists S \text{ atypical}\right] \leq \sum_{S \text{ is large}} \Pr\left[A_S\right] \leq 2^n \cdot \exp(-n) \approx 0.$$

This implies,

$$\Pr\left[\nexists S \text{ atypical}\right] \approx 1.$$

□

3 Expander Graphs

There are explicit constructions of such random-looking graphs with *bounded degree.* For example, there are several constructions based on algebraic / number-theoretic theorems (LPS, Margulis, ...)

For a long time it was not known how to construct an explicit graph that is an expander, through elementary methods. The zigzag construction, due to Reingold, Vadhan, and Wigderson, gave a way to construct an expander explicitly from elementary methods.

Below is a description based on Lecture 16 of Spielman's Spectral Graph Theory course.

Let G_0 be an expander on a constant number of vertices. This graph can be obtained by exhaustive search. We construct G_{i+1} from G_i through the following procedure:

1. Take the line graph of G_i (each vertex becomes a clique).

2. Replace the cliques by small expanders.

3. Square (twice).

The above procedure is repeated n times in order to obtain some $G_n = G$. For an analysis of the above construction see Lecture 16 of Spielman's Spectral Graph Theory course.

4 High Dimensional Expanders

We now extend our discussion of expanders to higher dimensions.

Let $X(0) = V$ be the set of vertices. Let $X(1) = E$ be the set of edges. We continue with $X(2)$ a set of (unordered) triples of vertices, sometimes called triangles. We keep going and have $X(d)$ be a set of so-called d-faces which are subsets of vertices of cardinality $d+1$. We denote by

$$X = X(0), X(1), \ldots, X(d)$$

X is a simplicial complex if whenever s is a face in X then every $s' \subset s$ is also a face in X.

We introduce the notion of a **link** of a vertex v as follows:

$$X_v = \{S \setminus \{v\} \mid v \in S \in X\},$$

This is analogous to the neighborhood of a vertex in a graph, and is itself a simplicial complex of one dimension less.

Actually, it makes sense to define the link of a face, not just of a vertex. The link of a face $s \in X$ is

$$X_s = \{S \setminus s \mid s \subset S \in X\},$$

Recall that a graph is a λ-(spectral)-expander if $\lambda(G) \leq \lambda$. We extend this definition recursively to a simplicial complex, as follows,

Definition 4.1. *A $\lambda-$skeleton expander is when:*

1. *The graph $G = (V = X(0), E = X(1))$ is an $\lambda-$expander.*

2. *$\forall v$, X_v is a $\lambda-$expander.*

Our first example of a high dimensional expander is the complete complex. Let us define it first:

Definition 4.2. *The complete complex $K_n(2)$ on n vertices is $X(0) = [n]$, $X(1) = \binom{[n]}{2} = \{\{ij\}\}_{i<j}$, and $X(2) = \binom{[n]}{3}$.*

Claim 4.3. *$K_n(2)$ is a $\lambda-$expander with $\lambda = O\left(\frac{1}{n-1}\right)$.*

Proof. It is easy to see that each link is a clique on $(n-1)$ vertices. □

We would like to now explore the following question: Is a random 2-dimensional complex an expander as defined above? We can answer this question in the Linial-Meshulam model: we insert each triangle (i.e., a 2-face) with probability p. In this model, we note that random 2-dimensional complex is an expander only when $p \geq 1/n$ (actually when $p \geq \frac{\log n}{n}$). Note that these random graphs are neither bounded degree nor sparse.

A sparse random 2-dimensional complex is not a high dimensional expander, as we will see in the exercise.

There exists a construction (due to Lubotzky, Samuels, and Vishne), (based on deep number theory and representation theory), of high dimensional graphs that are (sparse) bounded degree! More precisely, $\forall \lambda \exists$ such that for an infinite sequence of ns, there is a complex $X(0), X(1), X(2)$ such that $|X(2)| = O(|X(1)|)$.

We conclude the lecture with the following open question:

Open Question: Is there an (elementary) construction of expanders in high dimension that is analogous to the zigzag construction for dimension 1?