High Dimensional Expanders - Homework 1

Due: December 3, 2018

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTex. Please email your solution to `yotamd@weizmann.ac.il`.

1 Expander Graphs

Let G be a graph on n vertices. We denote the adjacency opertor of G by A_G, and we denote A_G's eigenvalues by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$.

1. Prove that **the second eigenvalue** $\lambda_2 = 1$ if and only if the graph is not connected. Prove that $\lambda_n = -1$ if and only if the graph is bipartite.

2. Compute the eigenvalues of G if G is the complete graph on n vertices (with the uniform measure on the edges).

3. Show that if G has no self loops, then $\lambda(A_G) \geq \frac{1}{n-1}$ (hint: use the expander mixing lemma, where $S, T = \{v\}$). Conclude that the complete graph is a graph with optimal two-sided expansion over all simple graphs on n vertices. Is this still true if we allow self loops?

2 The Zigzag Product

Complete the proofs for the claims given in the lecture:

1. Denote by P_G the walk on the blue edges, and $\widehat{A_H}$ the walk on red edges. Let $f : V_{zigzag} \to \mathbb{R}$ s.t. for any $v \in V_G$ and $i, j \in V_H$, $f(v,i) = f(v,j)$ (i.e. f is constant on clouds). Then

$$\langle P_G f, f \rangle = \langle A_G \tilde{f}, \tilde{f} \rangle$$

   where $\tilde{f} : V_G \to \mathbb{R}$ is defined by $\tilde{f}(v) = f(v,i)$ (the choice of i is arbitrary).

2. Let $f : G \textcircled{z} H \to \mathbb{R}$ be any function. Then we can orthogonally decompose $f = f^G + f^H$ s.t.

   (a) f^G is constant on any component $H_v := \{(v,i) : i \in H\}$.

   (b) For all $v \in G$, the expectation on each component H_v for f^H is 0, i.e

$$\underset{i \in H_v}{Ex}[F^H(v,i)] = 0.$$

   Furthermore, if $f \perp 1$ then $f^G \perp 1$.

For your convenience, we put here the constructions of the replacement product and the Zigzag product:

Definition 2.1 (The Replacement Product)**.** *Let G, H be graphs such that G has n vertices and is D regular, and H has D vertices. Suppose that for each vertex, the edges adjacent to the vertex are ordered (in any arbitrary order): $E_v = (e_1, ..., e_D)$. We treat the vertices of H as an index set $i = 1, ..., D$. The replacement product is the graph $G \textcircled{r} H = (V_{rep}, E_{rep})$ defined as follows:*

$$V_{rep} = V_G \times V_H,$$
$$E_{rep} = E_{rep}^{red} \cup E_{rep}^{blue},$$

where

$$E_{rep}^{red} = \{\{(v,i),(v,j)\} : \{i,j\} \in E_H\}$$

and

$$E_{rep}^{blue} = \{\{(v,i),(u,j)\} : e_i \in E_v,\ e_j \in E_u,\ e_i = e_j\}$$

according to the order chosen above.

Definition 2.2 (The Zigzag Product)**.** *Let G, H as above. The Zigzag product is the graph $G \textcircled{z} H$ whose vertex set is $V_{zigzag} = V_G \times V_H$, and edge set is*

$$E =$$

$$\{\{(v,i),(u,k)\} : \exists j_1, j_2\ \{(v,i),(v,j_1)\} \in E_{rep}^{red},\ \{(v,j_1),(u,j_2)\} \in E_{rep}^{blue},\ \{(u,j_2),(u,k)\} \in E_{rep}^{red},\}$$

The probability of $\{(v,i),(u,k)\}$ is the probability of walking from (v,i) to (u,k) in $G \textcircled{r} H$ in three steps, conditioned on taking the first and third steps in the red edges, and the second in the blue edges.

3 The Sparse Random Complex

Let $X(0)$ be a set of n vertices, and let d be a parameter. A triangle is a set of three distinct vertices in $X(0)$. For each $v \in X(0)$ let us choose d triangles containing v, uniformly and independently. Let $X(2)$ be the collection of at most nd triangles chosen in this way. Let $X(1)$ be all edges that participate in a triangle of $X(2)$.

1. Show that the probability of repetition, i.e. of two vertices choosing the same triangle, is small. Show that with high probability we have nd triangles.

2. Show that if $d \ll n$ then with high probability the link of a vertex v is highly disconnected. Describe how a typical link looks like. (Conclude that this is not a high-dimensional expander).

4 The Paths Complex

Let $G = (V, E)$ be a d-regular expander. We construct a 2 dimensional complex by setting $X(0) = V$, and constructing $X(2)$ by placing a triangle on every triple of distinct vertices u, v, w such that $uv, vw \in E$ (i.e. where uvw is a length-2 path). We let $X(1)$ be the edges of the triangles described above.

1. Show that $X(1) \supset E$. Who are the edges in $X(1) - E$?

2. Show that $|X(2)| = O(nd^2)$.

3. Prove that the link of each vertex $v \in X(0)$ is a connected graph.

4. The *girth* of a graph is the length of its shortest cycle. Assume that the girth of G is at least 5. Fix $v \in X(0)$ and let X_v be its link.

   - Give a full description of the link of a vertex v.
   - Recall that the edge expansion of a graph G is

   $$\Phi(G) = \inf_{\{S : 0 < \Pr[S] \leq \frac{1}{2}\}} \{\frac{\Pr[E(S, S^c)]}{\Pr[S]}\}.$$

   Find a set S of about half of the vertices in X_v that have only $O(|S|)$ outgoing edges (and not $\theta(d|S|)$). Deduce an upper bound on the edge expansion of X_v.[1]

Conclusion: This complex has connected links, which is better than the random complex. However, the expansion inside a link is weak in that it is not a constant *independent* of the number of vertices in the link. Such a graph is not considered an expander because in an expander the expansion is supposed to be independent of the number of vertices in the graph.

[1] You can read about Cheeger's inequality for expander graphs, and see how this also lower bounds the spectral expansion.