High Dimensional Expanders I

Lecture Notes: Intro and Link Expanders

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In this first lecture we begin with a short introduction to expander graphs. We then give a definition of high dimensional expanders (Definition 2.6) through the notion of links and simplicial complexes. We introduce and prove the trickling-down theorem of Oppenheim (Theorem 2.8) that shows how expansion trickles down from the links of a complex to its skeleton.

1 Expander Graphs

Recall, $G = (V, E)$ is a graph with set of vertices V and set of edges E. Throughout the course, we assume the edges are undirected, but could be weighted. We now give several definitions for expanders.

1.1 Combinatorial Definition

In this course we will talk about arbitrary graphs, However, in this subsection, we may assume that $G = (V, E)$ is a d-regular graph. The probability of a vertex set $S \subseteq V$ is

$$\Pr[S] = \frac{|S|}{|V|}.$$

The probability of a set of edges $F \subseteq E$ is defined in a similar manner:

$$\Pr[F] = \frac{|F|}{|E|}.$$

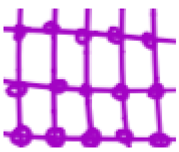
Expander graphs are graphs with no "bottlenecks". More Precisely, for $\emptyset \neq S \subset V$ with $\Pr(S) \leq \frac{1}{2}$ define

$$\phi(S) = \frac{\Pr(E(S, \bar{S}))}{\min(\Pr(S), \Pr[\bar{S}])},$$

where $E(S, \bar{S})$ is the set of edges between S and its complement $\bar{S} = V \setminus S$.

We say that G is an ε - edge expander if for all non-empty $S \subsetneq V$, $\phi(S) \geq \varepsilon$. In words, geometrically, this definition says that for every subset of vertices S the ratio of its boundary and its area is bounded from below. Namely, it has a large perimeter.

Non-Example 1.1 (Grid)**.** *Consider the following family $\{G_n\}$ of subgraphs of $\mathbb{Z}^2$. The vertices of G_n are $\{(i, j) : 0 \leq i, j \leq n\}$, i.e. vertices in a square of side length n. The edges connecting adjacent vertices as in the figure*[1].



[1] All figures taken from Irit Dinur's notes.

This family is not a family of expanders for any constant $\varepsilon > 0$. Indeed, for any n let S_n be the set of vertices in the form of a rectangle with side lengths n and $[\frac{n}{2}]$. Then S_n contains half the vertices of G_n, namely $O(n^2)$ vertices whereas the boundary of S_n contains $O(n)$ edges. Thus, the ratio tends to 0 as $n \to \infty$.

Example 1.2. *The complete graph on n vertices is an ε-edge expander where $\varepsilon = 1 - O\left(\frac{1}{n}\right)$.*

It is not trivial to construct a sequence of expander graphs where the degree of all vertices are uniformly bounded (as the number of vertices go to infinity). However, there are known constructions for such objects, both random and deterministic. We show one construction in the next lecture.

1.2 Probabilistic Definition

A graph is a distribution over (unordered) pairs. Every pair is assigned a probability mass. It is natural to define a random walk process with this distribution by: Given a vertex $v_i \in G$, choose v_{i+1}

1. Choose an edge $e \in E$ according to the distribution over edges, conditioned by $v_i \in e$. Denote $e = \{v_i, u\}$.
2. Choose $v_{i+1} = u$.

Question 1.3. *What is the probability distribution describing the t-th vertex in the random walk?*

Let $\pi_1 : E \to [0,1]$ denote the distribution on weighted unordered pairs (this is an equivalent way to describe the weighted graph). The distribution π_1, induces a measure on the vertices, $\pi_0 : V \to [0,1]$, given by $\pi(v) = \frac{1}{2}\sum_{e:v\in e} \pi_1(e)$ (we assume without loss of generality that the weights on the edges sum up to 1). It may be proven that when a graph is connected and isn't bipartite, the distribution of the t-th vertex converges to π_0 s t approaches infinity.

If G is an expander, the convergence rate to the stationary distribution π_0 is *rapid*. The convergence rate is $t = O(log(n))$, which is as fast as possible, when the degree of the graph is bounded and the number of vertices goes to infinity.

1.3 Algebraic Definition

Denote $\ell_2(V) = \{f : V \to \mathbb{R}\}$. The distribution defined above on the vertices pi_0 rise to an inner product on $\ell_2(V)$ defined by:

$$\langle f, g\rangle = \sum_{v\in V} \pi_0(v) f(v) g(v) = \mathbb{E}[f(v)g(v)].$$

The notions of orthogonality and norm are defined as usual:

$$f \perp g \Leftrightarrow \langle f, g\rangle = 0$$

$$\|f\|_2^2 = \langle f, f\rangle$$

We can view the graph as a linear operator on this space.

Definition 1.4 (The transition matrix)**.** *Given a graph $G = (V, E)$ with a probability distribution P on E (we assume that $P(e) \neq 0$ for all $e \in E$) we define the* transition matrix *A of G by*

$$A_{v,u} = Pr(u|v) = \frac{P(\{u, v\})}{\sum\limits_{\{w,v\}\in E} P(\{w, v\})}$$

The transition operator A acts on $f \in \ell_2(V)$ by:

$$Af(v) = \sum_{\{u,v\}\in E} A(u, v) f(u) = \mathop{\mathbb{E}}_{u|v}[f(u)],$$

where this expectation is over "u—v" which means that we choose a vertex $u \sim \pi_0$ conditioned on being adjacent to v.

Observe the following properties of A:

1. Denote the all-ones function by $\mathbb{1}$. This function in an eigenvector of A, with eigenvalue $\lambda_1 = 1$. In other words
$$A\mathbb{1} = \mathbb{1}.$$
2. All eigenvalues λ of A are $-1 \le \lambda \le 1$.
3. A is self adjoint. In fact, A corresponds to the following quadratic form:
$$\langle g, Af\rangle = \sum_{v\in V} \pi(v)g(v)Af(v) = \sum_{v\in V} \pi(v)g(v)\left(\sum_{uv\in E} Pr(u|v)f(u)\right) =$$
$$= \mathop{\mathbb{E}}_{(u,v)\in E}[g(v)f(u)] = \langle Ag, f\rangle$$

Corollary 1.5. *It follows that A is diagonalizable and so we have*
$$-1 \le \lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1 = 1.$$

We are now ready to give the spectral definition to expansion:

Definition 1.6 (Spectral Expander)**.** *A graph is a λ - two-sided spectral expander if* $\max(|\lambda_1|, |\lambda_2|) \le \lambda$*). A graph is a λ-one-sided spectral expander if $\lambda_2 \le \lambda$.*

The notion of a one-sided spectral expander is strictly weaker than the two-sided expander. For example, the complete bipartite graph is a one-sided expander for $\lambda = 0$, but not a two-sided expander, since its most negative eigenvalue is $\lambda_n = -1$.

The definitions we gave to expanders so far are "morally" equivalent. One example for a theorem that shows such a connection between the combinatorial and spectral definition is the Expander Mixing Lemma:

Lemma 1.7 (Expander Mixing Lemma)**.** *Let $G = (V, E)$ be any λ-two-sided spectral. Let $S, T \subset V$, and denote by $E(S, T)$ the set of directed edges that start with a vertex in S, and end with a vertex in T. Then*
$$|\Pr[E(S,T)] - \Pr[S]\Pr[T]| \le \lambda\sqrt{\Pr[S]\Pr[T](1-\Pr[S])(1-\Pr[T])}.$$

The proof of this lemma is given in the next lecture.

2 High Dimensional Hypergraphs

We would like to have an object that is a high dimensional analogue of a graph.

Definition 2.1. *A* hypergraph *is a pair (V, E) with V a set of vertices and E a set of subsets of V.*

Definition 2.2. *A* simplicial complex *(abbreviated s.c.) is a hypergraph that is downwards closed to containment. Namely, if $S \in E$ and $S' \subset S$ then $S' \in E$. $S \in E$ is called a face.*

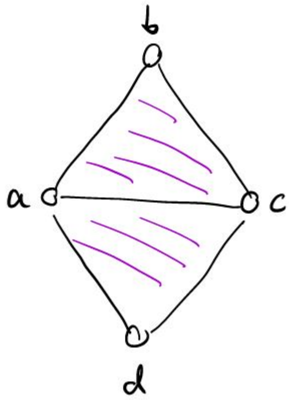
We usually partition a simplicial complex X to
$$X = X(0) \cup X(1) \cup \cdots \cup X(d)$$

where $X(i)$ is the set of faces of size $(i+1)$, or dimension i. In particular, $X(0)$ are identified with the vertices of the simplicial complex.

We say a simplicial complex is of *d-dimensional* if the maximal face size is $d+1$.

A d-dimensional simplicial complex is *pure* if all faces are contained in some (not necessarily unique) d-dimensional face.

Example: Consider the hypergraph below:



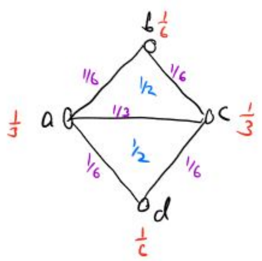
Then

$$X(2) = \{\{a,b,c\},\{a,c,d\}\}$$
$$X(1) = \{\{a,b\},\{b,c\},\{a,c\},\{b,c\},\{c,d\}\}$$
$$X(0) = \{\{a\},\{b\},\{c\},\{d\}\}$$

Definition 2.3. *Let X be a pure d-dimensional simplicial complex. We define on X measures $\pi_d, \pi_{d-1}, \dots, \pi_0$ as follows:*

- *π_d: an arbitrary distribution over $X(d)$*
- *π_i: The probability of choosing a face $T' \in X(i)$ is the probability of choosing a face $T \in X(d)$ with the distribution π_d and then choosing T' with uniform distribution over all faces in $X(i)$ that are contained in T.*

Continuing our example from above, weights $\pi_2(\{a,b,c\}) = \pi_2(\{a,c,d\}) = \frac{1}{2}$, the measure "trickles down" to π_1, π_0 as in the figure:



In a graph, it can be proven that the three definitions given above for expansion of a graph are "morally" equivalent. However, when we pass to a simplicial complex of dimension $d \geq 2$ those definitions generalize in different ways.

Now, we would like to extend our definition of expanding graphs to those new, high dimensional, geometric objects. Two questions naturally arise:

1. How to define expansion in higher dimensions?
2. How to construct expander graphs in high dimensions?

In today's lecture we focus on a local characterization of expansion. For this, we give the language we need to define expansion.

Definition 2.4 (Link)**.** *Let X be a d-dimensional simplicial complex and $S \in X(i)$. The link of S is a $(d-i-1)$-dimensional simplicial complex defined by:*

$$X_S = \{T - S | S \subset T \in X\}.$$

For example, in the figure above, the link of a is the graph $X_a(0) = \{b, c, d\}$ and $X_a(1) = \{\{b, c\}, \{c, d\}\}$.

Definition 2.5. *Let X be a simplicial complex and $k < d$ some non-negative integer. The k-skeleton of X is the subspace of X that is the union of faces of dimension $\leq k$.*

We are ready to define high dimensional expansion:

Definition 2.6 (λ-high Dimensional Expander)**.** *Let $\lambda < 1$. A d-dimensional pure simplicial complex X is a λ-two-sided high dimensional expander if:*

1. *The 1-skeleton of X is a two-sided λ-spectral expander, and*
2. *For any $i \leq d-2$ and all $s \in X(i)$ the 1-skeleton of X_s is a two-sided λ-spectral expander.*

There is also a weaker notion of *one-sided λ-high dimensional expanders*, where the 1-skeleton of the simplicial complex, and the 1-skeleton of all the links are *one-sided* λ-spectral expanders.

Example 2.7. *The d-dimensional complete complex on n vertices, which consists of all subsets of $\{1, ..., n\}$ of size $\leq d+1$, is an example for a two-sided high dimensional expander. The 1- skeleton of every link is a complete graph, which is a $\left(\frac{1}{n-d}\right)$-two-sided spectral expander (check!).*

Constructing bounded degree two-sided HDXs is challenging. For graphs, there are well known algebraic constructions, random constructions and combinatorial constructions (see next week's lecture), where the degree of every vertex is uniformly bounded (when the number of vertices goes to infinity).

For simplicial complexes, even 2-dimensional, the only known constructions are algebraic, see [**LubotzkySV2005a**], [**KaufmanO2017**].

A priori, this definition requires information on all links of the complex. However, the following theorem by Izhar Oppenheim [**Oppenheim2018**], tells us that if the links of the $(d-2)$-faces are good expanders, then the links of the lower dimension faces are also expanders, as long as they are connected. More precisely:

Theorem 2.8 (Trickling-Down Theorem, two-dimensional)**.** *Let X be a 2-dimensional simplicial complex such that the graph $(X(0), X(1))$ is connected and $\forall v \in X(0)$ X_v is a one-sided λ-expander. Then $(X(0), X(1))$ is a μ-expander where $\mu = \frac{\lambda}{1-\lambda}$.*

Note that the theorem is meaningless once $\lambda \geq \frac{1}{2}$.

By applying the theorem iteratively, we get the following useful corollary:

Corollary 2.9 (Trickling-Down Theorem, d-dimensional)**.** *Let X be a d-dimensional simplicial complex such that the 1-skeleton of every link (including the entire simplicial complex) is connected and $\forall v \in X(d-2)$ X_v is a one-sided λ-expander. Then X is a μ-expander where $\mu = \frac{\lambda}{1-(d-1)\lambda}$.*

Proof of Theorem 2.8. Let A be the adjacency operator associated with the 1-skeleton $(X(0), X(1))$.

Suppose $f : X(0) \to \mathbb{R}$ is an eigenfunction with eigenvalue γ, and assume $f \perp \mathbb{1}$. Also assume $\|f\| = 1$, namely $\mathbb{E}[f^2] = 1$. We have:

$$\gamma = \langle f, Af \rangle = \mathop{\mathbb{E}}_{\{u,w\} \in X(1)} f(u)f(w) = \mathop{\mathbb{E}}_{v \in X(0)} \mathop{\mathbb{E}}_{\{u,w\} \in X_v} [f(u)f(w)] \tag{2.1}$$

Next, let A_v be the adjacency operator associated with X_v. By assumption X_v is a one-sided λ-spectral expander and so the second largest eigenvalue of A_v satisfies $\lambda_2 \leq \lambda$.

For any function $g : X_v(0) \to \mathbb{R}$ satisfying $g \perp \mathbb{1}$ we have by the spectral decomposition of A that

$$\langle Ag, g \rangle \leq \lambda \|g\|^2.$$

Denote the restriction of f to a link X_v by f^v, namely:

$$f^v : X_v \to \mathbb{R}$$
$$f^v(u) = f(u).$$

Define $g^v = f^v - \gamma f(v)\mathbb{1}$. Recall that $Af(v) = \gamma f(v)$ and so we have $g^v \perp \mathbb{1}$ since

$$\mathop{\mathbb{E}}_{u \in X_v(0)}[f^v(u)] = Af(v) = \gamma f(v).$$

Now we evaluate

$$\mathop{\mathbb{E}}_{uw \in X_v}[f^v(u)f^v(w)] = \mathop{\mathbb{E}}_{v}[\langle f^v, A_v f^v \rangle] = \mathop{\mathbb{E}}_{v}[\langle g^v, A_v g^v \rangle] + \mathop{\mathbb{E}}_{v}(f(v)\gamma)^2 =$$
$$= \mathop{\mathbb{E}}_{v}[\langle g^v, A_v g^v \rangle] + \gamma^2$$

On the other hand, in 2.1 we can switch f to f^v since we only evaluate f on X_v and so we also have

$$\mathop{\mathbb{E}}_{v \in X(0)} \mathop{\mathbb{E}}_{uw \in X_v}[f^v(u)f^v(w)] = \mathop{\mathbb{E}}_{v \in X(0)} \mathop{\mathbb{E}}_{uw \in X_v}[f(u)f(w)] = \gamma$$

Therefore,

$$\gamma - \gamma^2 = \mathop{\mathbb{E}}_{v}[\langle g^v, A_v g^v \rangle] \leq \lambda \mathop{\mathbb{E}}_{v}[\|g^v\|^2] = \lambda(1 - \gamma^2)$$

We assumed that G is connected thus $\lambda_1 = 1$ has multiplicity 1, and we have $\gamma < 1$. and so we divide by $1 - \gamma$:

$$\gamma \leq \frac{\lambda}{1 - \lambda}.$$

□

Note that when $\lambda < \frac{1}{2}$ we have $\gamma \leq 2\lambda$. This theorem shows us that we can infer global properties of the graph based on local properties given by the links of the $(d-2)$-faces.