High Dimensional Expanders

Lecture Notes: Zig-Zag Product

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1 Reminder from Last Week

1.1 Weighted Graphs

Let $G = (V, E)$ be a weighted graph and $\Pi_1 : E \to [0,1]$ a distribution on the edges. We define the distribution on the vertices by

$$\Pi_0(v) = \frac{1}{2}\sum_{v\sim u}\Pi_1(\{v,u\}).$$

Denote the space of real valued functions with domain V by $\ell_2(V)$. This is an inner-product space where

$$\langle f, g\rangle = \mathop{\mathbb{E}}_{v\sim\Pi_0}[f(v)g(v)].$$

Example: In a k-regular graph, $\Pi_1(e) = \frac{1}{|E|}$ and $\Pi_0(v) = \frac{1}{|V|}$. This case is our main focus on today's talk.

1.2 The Adjacency operator

On any graph we have a an adjacency operator $A : \ell_2(V) \to \ell_2(V)$ that's defined by

$$Af(v) = \mathop{\mathbb{E}}_{u \underset{G}{\sim} v}[f(u)].$$

This operator corresponds to the symmetric bilinear form:

$$\langle Af, g\rangle = \langle f, Ag\rangle = \mathop{\mathbb{E}}_{(v,u)\in E}[f(v)g(u)].$$

(the orientation of an edge is chosen u.a.r.).

We saw that A is self-adjoint, thus it is *orthogonally diagonizeable* above the reals: i.e. there is a basis $(f_1, ..., f_{|V|})$ of $\ell_2(V)$ and scalars $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ s.t.:

1. $\forall i \neq j,\ f_i \perp f_j$.
2. $Af_i = \lambda_i f_i$.

Furthermore we can show that $\lambda_1 = 1$ and for all $1 \leq i \leq n$, $|\lambda_i| \leq 1$, and $\lambda_2 < 1$ iff the graph is connected.

1.3 Spectral Expander Graphs

Recall the definition from last time:

Definition 1.1 (Two-Sided Spectral Expansion)**.** *Let $G = (V, E)$ be a weighted graph, and let A_G be it's adjacency operator. Let $0 \leq \lambda < 1$. We say G is a λ-two-sided spectral expander if* $\max(|\lambda_2|, |\lambda_n|) \leq \lambda$.

It is convenient to denote by

$$\lambda(G) = \lambda(A_G) := \inf\{\lambda \leq 1 : \text{G is a } \lambda\text{-two-sided expander}\} = \left\| A|_{\{1\}^\perp} \right\|_2 .$$

Example: Let K_n be the complete graph on n vertices (weighted uniformly).

Exercise 1.2. *Show that K_n is a $\frac{1}{n-1}$-two-sided spectral expander.*
hint: *Consider K_n's adjacency operator's matrix form:*

$$\frac{1}{n-1}J - \frac{1}{n-1}I$$

where J is the all-ones matrix, and I is the identity. J has rank 1, so all eigenvalues of J, except the first one are 0.

Before discussing our construction for expander graphs, we take a detour to discuss a property of expander graphs:

The Expander Mixing Lemma

Recall that last week we saw that we can decompose any $f : V \to \mathbb{R}$ to $f = f^{\|} + f^{\perp}$, s.t. $f^{\|}$ is a constant function, and $f^{\perp}$ is orthogonal to the constant functions. This construction is respected by the graph operator A, i.e. $Af^{\|}$ is also constant, and $Af^{\perp}$ is also orthogonal to the constant functions. The following inequality is direct but useful $\left\|Af^{\perp}\right\| \leq \lambda(A) \left\|f^{\perp}\right\|$.

This decomposition is very useful, when we want to discuss how A operates on functions (e.g. Oppenheim's theorem from last time). Often (such as last time), our "meta proof" follows these lines:

1. Decompose a function $f = f^{\|} + f^{\perp}$ that has some combinatorial meaning.
2. Find some combinatorial meaning to $f^{\|}$.
3. Say that $Af^{\perp}$ is a negligible error.

We demonstrate this principle to prove this classical result, that show expander graphs have small discrepancy:

Lemma 1.3 (Expander Mixing Lemma)**.** *Let $G = (V, E)$ be a λ-two-sided spectral expander. Then for any two disjoint sets $S, T \subset V$ we have*

$$|Pr[E(S,T)] - Pr[S]Pr[T]| \leq \lambda\sqrt{Pr[S]Pr[T](1 - Pr[S])(1 - Pr[T])},$$

Where $Pr[E(S,T)]$ is the probability to choose an (oriented) edge $(s,t) \in E$ s.t. $s \in S, t \in T$.

When $T = V \setminus S$, the result of this lemma is the definition of edge expansion. However, this is stronger - this shows that the edges between *any* two disjoint sets S, T is proportionate to their respective sizes (as you'd expect in the complete complex), up to an error dominated by λ.

Proof. We want to bound something by the second eigenvalue of the adjacency operator, so it is reasonable to look for functions that represent S, T somehow, and look at what happens to them under the operation of A.

Consider the indicator functions $1_S, 1_T : V \to \mathbb{R}$, where $1_S(v) = 1 \iff v \in S$ (otherwise 0, resp. T). We can decompose

$$1_S = 1_S^{\|} + 1_S^{\perp},\ 1_T = 1_T^{\|} + 1_T^{\perp}$$

where $1_S^{\|}$ is the constant part of S, and $1_S^{\perp}$ is the part orthogonal to it.

Notice that

$$\langle A1_S, 1_T\rangle = \mathop{\mathbb{E}}_{(u,v)\in E}[1_S(u)1_T(v)].$$

$1_S(u)1_T(v)$ is one if and only if (u,v) is an edge between S,T thus

$$\mathop{\mathbb{E}}_{(u,v)\in E}[1_S(u)1_T(v)] = Pr[E(S,T)].$$

Now let us calculate $\langle A1_S, 1_T\rangle$ differently:

$$\langle A1_S, 1_T\rangle = \left\langle A1_S^{\parallel}, 1_T^{\parallel}\right\rangle + \left\langle A1_S^{\perp}, 1_T^{\perp}\right\rangle$$

or

$$\langle A1_S, 1_T\rangle - \left\langle A1_S^{\parallel}, 1_T^{\parallel}\right\rangle = \left\langle A1_S^{\perp}, 1_T^{\perp}\right\rangle.$$

We notice that $1_S^{\parallel}(v) = Pr[S]$, $1_T^{\parallel}(v) = Pr[T]$ (because $\mathbb{E}[1_S] = Pr[S]$). Thus

$$\left\langle A1_S^{\parallel}, 1_T^{\parallel}\right\rangle = Pr[S]Pr[T].$$

So we get

$$|\langle A1_S, 1_T\rangle - \left\langle A1_S^{\parallel}, 1_T^{\parallel}\right\rangle| = \frac{1}{2}|Pr[E(S,T)] - Pr[S]Pr[T]| \leq$$
$$|\left\langle A1_S^{\perp}, 1_T^{\perp}\right\rangle|.$$

Now we apply Cauchy-Schwartz to get:

$$|\left\langle A1_S^{\perp}, 1_T^{\perp}\right\rangle| \leq \left\|A1_S^{\perp}\right\| \left\|1_T^{\perp}\right\| \leq \lambda \left\|1_S^{\perp}\right\| \left\|1_T^{\perp}\right\| = \lambda\sqrt{Pr[S]Pr[T](1-Pr[S])(1-Pr[T])},$$

where the last equality is due to the fact that

$$\left\|1_S^{\perp}\right\|^2 = \|1_S\|^2 - \left\|1_S^{\parallel}\right\|^2 = Pr[S] - Pr[S]^2.$$

QED

□

2 Zigzag Product - An Infinite Familiy of Bounded Degree Expander Graphs

We already showed above that the sequence of graphs $\{K_n\}_{n=1}^{\infty}$ is a sequence of very good expanders. However, they are dense - $|E_n| = \Omega(n^2)$. **Our goal is to construct an infinite family of λ-two-sided spectral expanders $\{G_n\}_{n=1}^{\infty}$ for any $\lambda < 1$, with a growing number of vertices**.

In this lecture we construct this infinite family of bounded degree high dimensional expanders, through an iterative algorithmic procedure. This construction is due to Reingold, Vadhan and Wigderson [RVW00]. In this construction we have two steps (which we do over and over again):

1. The Powering step - where we take a graph and increase the spectral expansion, but also increase its degree.

2. The *Zigzag* product step - where we take a graph G and decrease its degree, and increase its number of vertices. The spectral expansion of the output graph gets decreased but using an auxiliary graph H, the decrease is mild.

$$G_0 \to G_0^2 \to G_1 = G_0^2 ⓩ H \to G_1^2 \to ...$$

Using these two steps we describe an infinite family of bounded degree expander graphs - where we start with a graph G_0 and an auxiliary graph H (with parameters we describe below). We define $G_{n+1} = G_n^2 ⓩ H$.

Remark 2.1. *A priori, the initial graphs are not known to exists, but one can prove their existence in a simpler manner (even take dense graphs, or use [Pin73] where the author proves random d-regular graphs are expanders when $d \geq 3$), and find them through exhaustive search. We will discuss this point later on, and just assume we have initial graphs.*

2.1 The Power Step

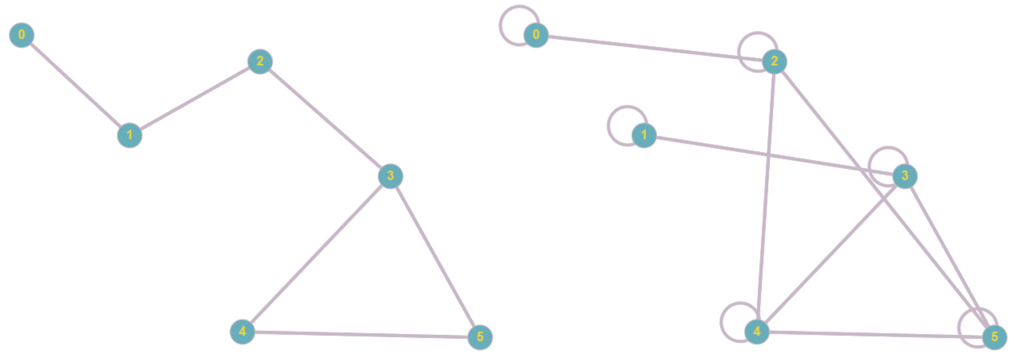
This step is the more trivial of the two.

Definition 2.2 (Powering of a Graph)**.** *Let $G = (V, E)$ be a graph. The powering of G, denoted by G^2 or $P(G)$ is the graph $G^2 = (V, E')$ where*

$$E' = \{\{x, z\} | \exists y \in V \ \{x, y\}, \{y, z\} \in E\}.$$

The probability of choosing an edge $\{x, z\}$, is the probability of choosing a 2-path in G from x to z.

Question 2.3. *Do we also include self loops (i.e. the multiset $\{x, x\}$)?*



Exercise 2.4. *Show that the adjacency operator of the power graph is $A_{G^2} = (A_G)^2$. Conclude that if G is a λ-two-sided spectral expander, then G^2 is a λ^2 spectral expander. Is this also true for a one-sided spectral expander?*

If we have a d-regular graph, the power G of it is a d^2-regular graph with d self-loops. We can remove our self loops, and the operator associated with the graph will be

$$\frac{d^2}{d^2 - d} \left[(A_G)^2 - \frac{1}{d} I \right].$$

(why?)

We won't go into the full analysis, but we can see that if G is close to an optimal d-degree expander graph, then so is G^2 (with or without removing self loops).

2.2 The Zig-Zag Product Graph

The *Zigzag* step is the main part of the construction. In this step we get a graph G on n vertices and degree D, and an auxiliary graph H with D vertices and degree d. Our output is a graph $G \textcircled{z} H$ with nD vertices and degree d. We will show that if the spectral expansion of G and H is good, then so is the spectral expansion of $G \textcircled{z} H$. Before describing the zigzag product, we begin by describing the replacement product:

Definition 2.5 (The Replacement Product)**.** *Let G, H be as above, and suppose that for each vertex, the edges adjacent to the vertex are ordered (in any arbitrary order): $E_v = (e_1, ..., e_D)$. We treat the vertices of H as an index set $i = 1, ..., D$. The replacement product is the graph $G \textcircled{r} H = (V_{rep}, E_{rep})$ defined as follows:*

$$V_{rep} = V_G \times V_H,$$

$$E_{rep} = E_{rep}^{red} \cup E_{rep}^{blue},$$

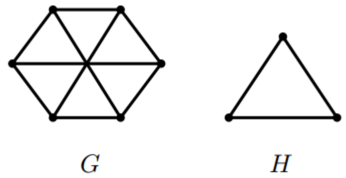
where

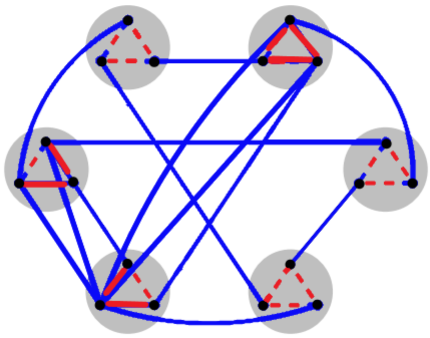
$$E_{rep}^{red} = \{\{(v,i),(v,j)\} : \{i,j\} \in E_H\}$$

and

$$E_{rep}^{blue} = \{\{(v,i),(u,j)\} : e_i \in E_v,\ e_j \in E_u,\ e_i = e_j\}$$

according to the order chosen above.[1]





$G \textcircled{r} H$

Pictorially, imagine the following: we have a graph G with degree D and n vertices. Now, we replace each vertex in G with a cloud of D vertices - the vertices in H.

The blue edges are the edges between indexes (i.e. if v, u are adjacent in G, then there are two distinct vertices in the cloud $(v,i),(u,j)$ that are adjacent with a blue edge.

The red edges are just local copies of the graph H.

We won't give a probabilistic description to the random walk on $G \textcircled{r} H$, but we will give two random walks - one on the blue edges, and one on the red.

1. On the red edges - the probability for each edge is $Pr[(v,i),(v,j)] = Pr_G[v]Pr_H[(i,j)]$.

2. One the blue edges $Pr[(v,i),(u,j)] = Pr_G[(v,u)]$.

Note that both walks are not even connected. However, composing them one after another gives the zigzag product.

Definition 2.6 (The Zigzag Product)**.** *Let G, H as above. The Zigzag product is the graph $G \textcircled{z} H$ whose vertex set is $V_{zigzag} = V_G \times V_H$, and edge set is*

$$E = \{\{(v,i),(u,k)\} : \exists j_1, j_2\ \{(v,i),(v,j_1)\} \in E_{rep}^{red},\ \{(v,j_1),(u,j_2)\} \in E_{rep}^{blue},\ \{(u,j_2),(u,k)\} \in E_{rep}^{red},\}.$$

The probability of $\{(v,i),(u,k)\}$ is the probability of walking from (v,i) to (u,k) in $G \textcircled{r} H$ in three steps, conditioned on taking the first and third steps in the red edges, and the second in the blue edges.

The reason for the name "Zigzag" is the red-blue-red walk.

[1] Picture adapted from lecture by Neil Olver.

3 Intuition for Why the Spectral Expansion is Good

Think about the walk on G. Suppose we traversed to $v \in V_G$ by the edge $\{v, w\}$. In the next step we choose one edge $\{v, u\} \in E_v$ uniformly at random (and independent of $\{v, w\}$), and traverse it to an edge u.

In the *Zigzag* product, after we traversed to v by $\{v, w\}$, we choose the next edge $\{v, u\}$, but not independent of $\{v, w\}$ - we choose it as a step in the auxiliary expander graph H.

Our intuition is that a step in an expander graph is *"similar"* (in some sense) to choosing independently. Thus our product graph, is the derandomization of the traversing process in G, by the expander H. If G, H are good enough expanders, we should get a decent expander by this.

4 Quick Analysis

It is easy to see that the degree of $G ⓩ H = degree(H)^2$, no matter the initial degree of G. Thus we achieve a degree reduction. In addition, we increase the number of vertices multiplicatively by $degree(G)$. It is left to analyze the spectral expansion of the resulting graph.

First of all, notice that separately, the red-edge graph, or the blue edge graph are bad in terms of expansion. However, there are some functions where the adjacency operators of the red, or blue edge graphs operate well:

Claim 4.1. *Denote by P_G the walk on the blue edges, and $\widehat{A_H}$ the walk on red edges.*

1. *Let $f : V_{zigzag} \to \mathbb{R}$ s.t. for any $v \in V_G$ and $i, j \in V_H$, $f(v, i) = f(v, j)$ (i.e. f is constant on clouds). Then*

$$\langle P_G f, f \rangle = \left\langle A_G \tilde{f}, \tilde{f} \right\rangle$$

   where $\tilde{f} : V_G \to \mathbb{R}$ is defined by $\tilde{f}(v) = f(v, i)$ (the choice of i is arbitrary).

2. *Left $g : V_{zigzag} \to \mathbb{R}$ be s.t. for any $v \in G$,*

$$\mathop{\mathbb{E}}_{i \in V_H} [g(v, i)] = 0.$$

   Then

$$\left\| \widehat{A_H} g \right\| \leq \lambda(A_H) \left\| g \right\|.$$

Proof. The first item is left as an exercise [demonstrate on board].

As for the second item, Notice that the red-edge walk is separated to connected components, hence when calculating

$$\left\| \widehat{A_H} g \right\|^2 = \left\langle \widehat{A_H} g, \widehat{A_H} g \right\rangle = \mathop{\mathbb{E}}_{((v,i),(v,j)) \in G^2_{E^{red}}} [g(v, i) g(v, j)],$$

we can use conditional expectation on $v \in G$ (i.e. first calculate the average for each component H_v, and then average them separately).

$$\mathop{\mathbb{E}}_{((v,i),(v,j)) \in E^{red}} [g(v, i) g(v, j)] = \mathop{\mathbb{E}}_{v \in G} [\langle A_H g|_{H_v}, A_H g|_{H_v},] \rangle.$$

In each component H_v, $\|A_H g|_{H_v}\| \leq \lambda(A_H) \|g|_{H_v}\|$ (because g on each component is orthogonal to the constant function) thus from Cauchy-Schwartz the result follows:

$$\mathop{\mathbb{E}}_{v \in G} [\langle A_H g|_{H_v}, A_H g|_{H_v},] \rangle \leq \mathop{\mathbb{E}}_{v \in G} [\lambda(A_H)^2 \langle g|_{H_v}, g|_{H_v},] \rangle = \lambda(A_H)^2 \langle g, g \rangle.$$

$\square$

Equipped with the claim above, we begin by decomposing functions on the vertices in a manner that respects the red and blue edges.

Claim 4.2 (Decomposition of functions)**.** *Let $f : G \textcircled{z} H \to \mathbb{R}$ be any function. Then we can orthogonally decompose $f = f^G + f^H$ s.t.*

1. *f^G is constant on any component $H_v := \{(v, i) : i \in H\}$.*
2. *For all $v \in G$, the expectation on each component H_v for f^H is 0, i.e*

$$\mathop{\mathbb{E}}_{i \in H_v}[f^H(v, i)] = 0.$$

Furthermore, if $f \perp 1$ then $f^G \perp 1$.

Proof. The full proof is an exercise. We note however that to obtain the value of f^G in each cloud $v \in G$, we can simply average f on the vertices of the cloud. □

This decomposition of the functions on the vertex space, corresponds to the decomposition of the adjacency operator of the zigzag product as follows:

Claim 4.3 (Decomposition of operators)**.** *We can decompose the adjacency operator M of $G \textcircled{z} H$ to*

$$M = \widetilde{A_H} P_G \widetilde{A_H}$$

where

1. *$\tilde{A_H}$ is a step on the red edges.*
2. *P_G is a step on the blue edges.*

Proof. Also an exercise (immediate from the definition). □

Given these claims, we prove a bound on the spectral expansion of the zigzag product. There is a more delicate analysis on the graph that gives a tighter bound, however we give this result only.

Theorem 4.4. *Let G be an α-two-sided spectral expander and H be a β-two-sided spectral expander. Then $G \textcircled{z} H$ is a $(\beta + \max(\alpha, \beta^2))$-two-sided spectral expander.*

Proof. Let $f : V_G \times V_H \to \mathbb{R}$ be a function on the vertices that is perpendicular to the constant functions. Denote by M the adjacency operator of the zigzag-product. By our two previous claims

$$\langle Mf, f\rangle = \left\langle Mf^G, f^G\right\rangle + \left\langle Mf^H, f^H\right\rangle + 2\left\langle Mf^G, f^H\right\rangle =$$

$$\left\langle \widetilde{A_H} P_G \widetilde{A_H} f^G, f^G\right\rangle + \left\langle \widetilde{A_H} P_G \widetilde{A_H} f^H, f^H\right\rangle + 2\left\langle \widetilde{A_H} P_G \widetilde{A_H} f^G, f^H\right\rangle \tag{4.1}$$

It is immediate that $\widehat{A_H} f^G = f^G$. Thus by item 1 of the first claim we obtain that

$$\left\langle \widetilde{A_H} P_G \widetilde{A_H} f^G, f^G\right\rangle = \left\langle P_G \widetilde{A_H} f^G, \widetilde{A_H} f^G\right\rangle = \left\langle P_G f^G, f^G\right\rangle \leq \alpha \left\|f^G\right\|^2.$$

Given the second item of the first claim, and Cauchy-Schwartz:

$$\left\langle \widetilde{A_H} P_G \widetilde{A_H} f^H, f^H\right\rangle \leq \|P_G\| \left\|A_H f^H\right\|^2 \leq \beta^2 \left\|f^H\right\|^2 = \beta^2(\|f\|^2 - \left\|f^G\right\|^2),$$

and

$$2\left\langle \widetilde{A_H} P_G \widetilde{A_H} f^G, f^H\right\rangle \leq 2\beta \left\|f^G\right\| \left\|f^H\right\| \leq \beta \|f\|^2.$$

The theorem follows. □

Open Question

This construction is an iterative algorithmic construction for expander graphs. This gives rise to the following open question:

Question 4.5. *Is there a similar construction where we can get two high dimensional expanders, and produce another high dimensional expander?*

References

[Pin73] Mark S. Pinsker. "On the complexity of a concentrator". In: *7th International Teletraffic Conference*. 1973.

[RVW00] Omer Reingold, Salil Vadhan, and Avi Wigderson. "Entropy waves, the zig-zag graph product, and new constant degree expanders and extractors". In: *Proceedings of the 41st FOCS*. 2000, pp. 3–13.