In this lecture we present random walk on graphs 2.1, then we generalize to random walks on high dimensional complexes 2.2 and introduce the lazy and non-lazy variant of random walks. We end with stating that random walk expander is a two-sided spectral expander 2.3.

1 The “ideal” expander

Let $G$ a graph with eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$ then

- $G$ is a $\lambda$ one-sided spectral expander if $\lambda_2 \leq \lambda$
- $G$ is a $\lambda$ two-sided spectral expander if $\max(|\lambda_2|, |\lambda_n|) \leq \lambda$

The “best” or rather “ideal” expander is a complete graph with self loops. For such a complete graph, the eigenvalues are $1 = \lambda_1 \geq \lambda_2 = \ldots = \lambda_n = 0$, and the appropriate adjacency matrix $J$ is

$$J = \begin{bmatrix}
\frac{1}{n} & \frac{1}{n} & \cdots \\
\frac{1}{n} & \cdot & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix} = \frac{1}{n} \mathbb{I}_{n \times n}.$$

One can see that $J \cdot \mathbb{1} = \mathbb{1}$.

And that for $f \perp \mathbb{1}$, $J \cdot f = 0 \cdot f$.

We have also previously defined $\langle f, g \rangle = \mathbb{E}_{v \sim \pi} f(v) \cdot g(v)$ where $\pi_0$ is the distribution over vertices.

And the matrix $J_\pi = \begin{bmatrix}
- & \pi_0 \\
- & - \\
\vdots & \vdots \\
- & -
\end{bmatrix}$

For $f \perp \mathbb{1}$ it holds that $\langle f, \mathbb{1} \rangle = 0$ namely $\mathbb{E}_{v \sim \pi} f(v) = 0$ i.e. $J_\pi \cdot f = \sum_{v \in V} \pi_0(v) \cdot f(v) = 0$

2 Random walks

**Definition 2.1.** Let $M : \ell_2(V_1) \to \ell_2(V_2)$ be a linear operator $\|M\|_{op} \triangleq \sup_{f \neq 0} \frac{\|Mf\|_{\ell_2(V_2)}}{\|f\|_{\ell_2(V_1)}}$

Note: sometimes people study transformation between $\ell_p$ to $\ell_q$, we focus only on $\ell_2$

**Claim 2.2.** Suppose $M$ is self adjoint, then $\|M\| = |\lambda_{max}|$

**Proof.** $M$ is self adjoint, therefore, it has an orthogonal basis of eigenfunctions. i.e. for every $f$ we can write $f = \sum_{i=1}^{n} \alpha_i g_i$, where $g_i$s are basis of eigenfunctions/eigenvectors

$$\|M\|^2 = \langle Mf, Mf \rangle = \sum \alpha_i \lambda_i \alpha_i \lambda_i \leq \left( \sum \alpha_i^2 \right) \cdot \lambda_{max}^2.$$

$$\|f\|^2 = \langle f, f \rangle = \sum \alpha_i^2.$$ 

\[ \square \]
An expander graph can be viewed as a linear operator that approximates $J$ in the operator norm.

**Claim 2.3.** If $A$ is the transition matrix of a $\gamma$-spectral-expander then $\|A - J\| \leq \gamma$. In particular, for all $f$ 

$$\|(A - J)f\| \leq \gamma \cdot \|f\|$$

**Proof.** Let $f_i$ be the eigenfunction of $A$ corresponding the the $i$th eigenvalue $\lambda_i$. Clearly $f_1 = 1$. Notice $A1 = 1$ and $J1 = 1$ thus $(A - J)1 = 0$. Moreover, for $i > 1$, $f_i \perp 1$: 

$$\|Af_i\| \leq \gamma f_i$$

and 

$$Jf_i = 0$$

therefore $(A - J)f_i = Af_i - 0 = \lambda_i f_i$. We get 

$$(A - J)f_i = \begin{cases} 0 & i = 1 \\ \lambda_i f_i & i \geq 1 \end{cases}$$

which implies 

$$\|A - J\| = \lambda_{max}(A - J) = max\{\|\lambda_2(A)\|, |\lambda_n(A)|\} \leq \gamma$$

\[\Box\]

### 2.1 Random walks on graphs

A random walk on a graph $G = (V, E)$ is a sequence (either finite or infinite), of vertices from $V$ (e.g $(v_0, v_1, v_2, v_1, \ldots)$). There are many types of random walks with different properties. One such type of random walk can be described by the following algorithm:

**Definition 2.4 (Up-down random walk).** On graph $G$

- **Start at a random vertex** $v_0 \sim \pi$
  
  Suppose we are at some vector $v_t$. Choose $v_{t+1}$ by the following process:

- **UP:** Choose a random edge $e$ containing $v_t$

- **DOWN:** Choose a random vertex $u \in e$ (uniformly): $v_{t+1} = u$

**Notice:** there is always probability $1/2$ to stay in place, which makes the transition matrix of the up-down random walk to be 

$$T_{UP-DOWN} = \frac{1}{2}Id + \frac{1}{2}A$$

where $A$ is the transition matrix of the non-lazy up-down random walk (where in the down step we chose $v_{t+1} \neq v_t$).

To ease our understanding of random walks, consider a bipartite graph $B(G)$ with two parts $V_0, V_1$ such that $V_0 = V_G, V_1 = E_G$. i.e. a bipartite graph that is made of a set of vertices $V_0$ which represents vertices in $G$ and a set of vertices $V_1$ which represents edges in $G$. In the bipartite graph $B(G)$ there will be edges between vertices $v \in V_0, e \in V_1$ if $v \in e$. i.e. $E_M = \{(v, e) \text{ if } v \in e \text{ for } v \in V_G, e \in E_G\}$. Notice that the the vertices in $V_1$ will be 2-regular.

![Example of a graph G and its appropriate graph B(G)](image-url)
Looking more carefully, we can break the random walk into two steps that are being executed alternately over and over again. These are the selection of edge (up) and the selection of a vertex in the edge (down). Which leads us to define and examine the following operators:

**Definition 2.5.** The Down operator $D : \ell_2(X(1)) \to \ell_2(X(0))$ is defined by

$$Df(v) \triangleq \mathbb{E}_{e|v} f(e)$$

(recall our notation $e|v$ is an edge $e$ which contains $v$, $v|e$ is a vertex $v$ contained in $e$). More generally $s'|s$ is a face $s'$ that is either containing $s$ or contained in $s$.

The Down operator is a linear operator, it may be helpful to think of the matrix form:

\[
\begin{pmatrix}
X(0) \\
X(1)
\end{pmatrix}
\begin{pmatrix}
D
\end{pmatrix}
\begin{pmatrix}
f
\end{pmatrix}
\]

Notice that $D[e, v] = \Pr[e|v]$ and that $D$ is row-stochastic, i.e. $\sum_{e \in E} D[e, v] = 1$.

**Definition 2.6.** The Up operator $U : \ell_2(X(0)) \to \ell_2(X(1))$ is defined by

$$Ug(e) = \mathbb{E}_{v|e} g(v) = \mathbb{E}_{v|v_1, v_2} g(v) = \frac{1}{2} g(v_1) + \frac{1}{2} g(v_2)$$

as there are exactly two (uniform) choices to choose a vertex from an edge.

The up-down random walk we have defined in Definition 2.4 is actually a composition of operators:

\[
\ell_2(V) \xrightarrow{U} \ell_2(E) \xrightarrow{D} \ell_2(V)
\]

\[
DU : \ell_2(V) \to \ell_2(V)
\]

\[
DUg(v) = \mathbb{E}_{v'|v} G(V) = \mathbb{E}_{e|v} G(V) = \frac{1}{2} Id + \frac{1}{2} A
\]

(2.1)

Where $A$ is the adjacency matrix of the graph. One may think of this as a two step walk on the bipartite graph $B(G)$ starting from the $v$ side.

### 2.2 Random walks in higher dimension

Let us now introduce the higher dimensional version of the up-down random walk. From now on, this will be referred to as the upper random walk.

**Definition 2.7** (Upper random walk). For $k \leq d - 1$

- **Start at a random $k$-face $s \sim \pi_k$**
  
  Suppose at time $t$ we are at some face $s_t$. Choose $s_{t+1}$ by the following process:

- **UP**: Choose $s'$ by choosing uniformly random $(k+1)$-face that contains $s_t$.

- **DOWN**: Choose $s_{t+1}$ to be a $k$-face by choosing uniformly random a single vertex from face $s'$ and erase it ($s_{t+1} \subset s'$).
Let us now introduce the higher dimensional version for the up and down operators:

- **Down operator** $D : \ell_2(X(i+1)) \to \ell_2(X(i))$ is defined by $Df(s) \triangleq \mathbb{E}_{s'|s} f(s')$ where $s \subset s'$ i.e. the expectation is conditioned on $s'$ that contains $s$.

- **Up operator** $U : \ell_2(X(i)) \to \ell_2(X(i+1))$ is defined exactly the same but for $s' \subset s$ by $Uf(s) \triangleq \mathbb{E}_{s'|s} f(s')$ i.e. the expectation is conditioned on $s'$ that is a subset of $s$.

Having the down operator generalized for higher dimensions, we can also introduce another type of random walk which is defined as applying the down operator before the up operator.

**Definition 2.8 (Lower random walk).** for $d \geq k \geq 0$

- Start at random $k$-face $s_0 \sim \pi_k$
  Suppose at time $t$ we are at some face $s_t$. Choose $s_{t+1}$ by the following process:
  - **DOWN**: Choose $s'$ to be a $(k-1)$-face by choosing uniformly random single vertex from face $s_t$ and remove it.
  - **UP**: Choose $s_{t+1}$ by choosing a random $(k)$-face that contains $s_t$ ($s' \subset s_{t+1}$, according to $\pi_k|s_t$)

Notice: Performing the lower random walk with $i = 0$ is going down to $X(-1) = \{\emptyset\}$. Performing the up step from the empty set is equivalent to choosing some vertex $v \sim \pi_0$, i.e. selecting a vertex distributed according to the stationary distribution. This means that $v_{t+1}$ is chosen independently from $v_t$. This is nothing but $J_{\pi}$.

We have just defined the upper random walk and the lower random walk for any simplicial complex.

**How do the upper random walk and lower random walk relate to each other?**

Example: Let us compute the random walk of $DU$ and $UD$ on a 1-dimensional complex (i.e. a graph). **Reminder:**

$$UDf = \begin{bmatrix} - & \pi_0 & - \\ - & \vdots & - \end{bmatrix} f$$

$$DUf = \left(\frac{1}{2}Id + \frac{1}{2}A\right) f$$

where $A$ is the transition matrix of the graph. Notice that for the upper random walk ($DUf$) there is a $1/2$ probability to stay at the same vertex (starting at vertex $v$, choosing an edge, and then a vertex in the edge, has probability $1/2$ to choose $v$). Such a behavior is called $1/2$-lazy random walk. We’ve seen Equation 2.3 as a result of Equation 2.1 and the analysis that followed.

**Definition 2.9 (Non-lazy upper random walk).** The non-lazy upper random walk has 0 probability to return to same face/vertex, i.e. for $k \leq d - 1$

- Start at a random $k$-face $s \sim \pi_k$
  Suppose at time $t$ we are at some face $s_t$. Choose $s_{t+1}$ by the following process:
  - **UP**: Choose $s'$ by choosing a random $(k+1)$-face that contains $s_t$ according to $\Pi_{k+1}|s_t$
  - **DOWN**: Choose $s_{t+1}$ to be a $k$-face by choosing uniformly at random a single vertex from face $s'$ and erase it ($s_{t+1} \subset s'$) $s.t. s_{t+1} \neq s_t$

i.e.

$$A = 2DU - ID$$

Where $A$ is the relevant adjacency matrix of the graph. This derives directly from Equation 2.3.
Describing the lazy upper random walk on faces in $X(k)$ we get that $DU = \frac{k+1}{k+2} A_k + \frac{1}{k+2} Id$ where $A_k$ is a transition matrix between faces in $X(k)$. From this formulation we can derive algebraic description for the non-lazy upper random walk $A_k = \frac{k+2}{k+1} DU - \frac{1}{k+1} Id$

Let $M^+_k$ denote the non-lazy upper random walk transition operator ($A_k$ from above). If $k$ is clear from the context, we omit it and write $M^+$.

### 2.3 Random walks on spectral $\gamma$-bounded high dimensional expander

We have seen in Claim 2.3 that in expander graphs

$$\| J_\pi - A \| \leq \gamma.$$  

We will now describe random walks on complexes to define expansion of complexes.

**Notations:**

- Down operator from $X(k+1)$ to $X(k)$: $D_{\searrow k} = D_{k+1 \searrow}$
- Up operator to $X(k)$ from $X(k-1)$: $U_{\nearrow k} = U_{k-1 \nearrow}$
- $M^+_k$ non-lazy upper random walk moving from $X(k)$ to $X(k)$

(When the context is clear we may omit the index $k$)

**Claim 2.10.** \( \langle U_{\nearrow k+1} g, f \rangle_{X(k+1)} = \langle g, D_{\searrow k} f \rangle_{X(k)} \)

**Proof.**

$$\langle U_{\nearrow k+1} g, f \rangle_{X(k+1)} = \mathbb{E}_{s \in \pi_{k+1}} [U_{\nearrow k+1} g(s) \cdot f(s)]$$

$$= \mathbb{E}_{s \in \pi_{k+1}} g(s') \cdot f(s)$$

$$= \mathbb{E}_{s \in \pi_{k+1}} f(s) \cdot g(s')$$

$$= \mathbb{E}_{s \in \pi_{k+1}} D f(s') \cdot g(s')$$

$$= \langle g, D_{\searrow k} f \rangle_{X(k)}$$

**Definition 2.11** (\( \gamma \) random walk high dimensional expander). A $d$-dimensional complex $X$ is a $\gamma$ random walk high dimensional expander if for every $k$ s.t. $0 \leq k \leq d - 1$

$$\| M^+_k - U_{k \nearrow} D_{k \searrow} \| \leq \gamma$$

i.e. a complex is $\gamma$-random-walk high dimensional expander if the distance between the non-lazy upper random walk and the lower random walk is bounded by $\gamma$ for every dimension $0 \leq k \leq d - 1$.

**Lemma 2.12.** If $X$ is a $d$-dimension $\gamma$-two-sided-link expander then it is a $\gamma$-random-walk expander

**Recall:** $X_s$ is the link of $s$ where $s$ is some face in $X$.

**Recall:** $X$ is a $\gamma$-link expander if for every $s \in X(k)$, $-1 \leq k \leq d - 2$ : $\lambda_{max}(X_s) \leq \gamma$ (in this notation we refer to the maximal absolute eigenvalue ($\lambda_{max} = max(|\lambda_2|, |\lambda_n|$)) of the 1-skeleton of $X_s$).
Proof. We will prove the lemma for \(d = 2\).

for \(k = 0\):

We need to show: \(\|M^+_0 - U \gamma_0 D_{0^}\| \leq \gamma\)

\(\|A - J_x\| \leq \gamma\) (By definition of expansion of the link of \(\emptyset\))

This follows since \(M^+_0\) is the non lazy upper random walk, (i.e. \(M^+_0 = 0 \cdot Id + 1 \cdot A = A\)) and \(U \gamma_0 D_{0^}\) is the lower random walk to \(X(-1)\) which the operator \(J_x\) introduced previously

for \(k = 1\): We need to show \(\|M^+_1 - U \gamma_1 D_1\| \leq \gamma\)

It is enough to show \(|\langle (M^+_1 - UD) f, f \rangle| \leq \gamma|f, f\rangle\).

For each \(v \in X(0)\) we define \(f^v : X_v(0) \rightarrow \mathbb{R}\) as a localization of \(f\) to vertices in the link of \(v\). i.e. for \(u \in X_v(u)\) \(f_v(u) := f((v, u))\).

\[
\langle M^+_1 f, f \rangle = \mathbb{E}[M^+_1 f(e) \cdot f(e) = \mathbb{E}_{e \sim e'} f(e) \cdot f(e')
\]

\[
= \mathbb{E}_{e \in X(2)} \mathbb{E}_{e_1 \neq e_2 \subseteq e} f(e_1) \cdot f(e_2)
\]

\[
= \mathbb{E}_{v} \mathbb{E}_{v \sim (u, v, w)} f(e_1) \cdot f(e_2)
\]

\[
= \mathbb{E}_{v} \mathbb{E}_{v \sim (0, v, w)} f^v(\{u, v\} \setminus v, f^v(\{w, v\} \setminus v)
\]

\[
= \mathbb{E}_{v} \mathbb{E}_{u, w \sim X_v(0)} f^u(v) f^w(v)
\]

To summarize, let \(f\) be an eigenfunction of eigenvalue \(\lambda_i\) for \(i \geq 2\) s.t. \(\|f\| = 1\). As such \(f \perp 1\) and so \(Jf = 0\) (As \(Jf = 0\) for all \(f \perp 1\) ).

\[
|\langle M^+_1 f, f \rangle - \langle UD f, f \rangle| = \mathbb{E}_{v} \mathbb{E}_{u, w \sim X_v(1)} f^u(v) f^w(w) - \mathbb{E}_{u, w \sim X_v(0)} f^u(v) f^w(w)
\]

\[
= \mathbb{E}_{v} \mathbb{E}_{u, w \sim X_v(1)} f^u(v) f^w(w) - \mathbb{E}_{u, w \sim X_v(0)} f^u(v) f^w(w)
\]

\[
= \mathbb{E}_{v}[|f^v| - \langle J_v f^v, f^v \rangle]
\]

\[
\leq \mathbb{E}_{v} \|f^v\|^2 = \gamma \mathbb{E}_{v} \|f^v\|^2
\]

The last inequality follows Claim 2.3 because the link of \(v\) is an expander graph, as \(X\) is a \(\gamma\)-link-expander. \(\square\)