High Dimensional Expanders

Lecture 3: Random Walks on High Dimensional Expanders

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In this lecture we present random walk on graphs 2.1, then we generalize to random walks on high dimensional complexes 2.2 and introduce the lazy and non-lazy variant of random walks. We end with stating that random walk expander is a two-sided spectral expander 2.3.

1 The "ideal" expander

Let G a graph with eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$ then

- G is a λ one-sided spectral expander if $\lambda_2 \leq \lambda$
- G is a λ two-sided spectral expander if $max(|\lambda_2|, |\lambda_n|) \leq \lambda$

The "best" or rather "ideal" expander is a complete graph with self loops. For such a complete graph, the eigenvalues are $1 = \lambda_1 \geq \lambda_2 = ...\lambda_n = 0$, and the appropriate adjacency matrix J is

$$J = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots \\ \frac{1}{n} & \ddots & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \frac{1}{n} \cdot \mathbb{1}_{n\times n}.$$

One can see that $J \cdot \mathbb{1} = \mathbb{1}$.
And that for $f \perp \mathbb{1}$, $J \cdot f = 0 \cdot f$.

We have also previously defined $\langle f\,, g\rangle = \mathbb{E}_{v\sim\pi} f(v) \cdot g(v)$ where π_0 is the distribution over vertices.
And the matrix $J_\pi = \begin{bmatrix} - & \pi_0 & - \\ - & \vdots & - \end{bmatrix}$

For $f \perp \mathbb{1}$ it holds that $\langle f\,, \mathbb{1}\rangle = 0$ namely $\mathbb{E}_{v\sim\pi} f(v) = 0$ i.e. $J_\pi \cdot f = \sum_{v\in V} \pi_0(v) \cdot f(v) = 0$

2 Random walks

Definition 2.1. *Let $M : \ell_2(V_1) \to \ell_2(V_2)$ be a linear operator $\|M\|_{op} \triangleq sup_{\substack{f\neq 0 \\ f\in l_2(V_1)}} \frac{\|Mf\|_{\ell_2(V_2)}}{\|f\|_{\ell_2(V_1)}}$*

Note: sometimes people study transformation between ℓ_p to ℓ_q, we focus only on ℓ_2

Claim 2.2. *Suppose M is self adjoint, then $\|M\| =| \lambda_{max} |$*

Proof. M is self adjoint, therefore, it has an orthogonal basis of eigenfunctions. i.e. for every f we can write $f = \sum_{i=1}^{n} \alpha_i g_i$ where g_is are basis of eigenfunctions/eigenvectors

$$\|M\|^2 = \langle Mf\,, Mf\rangle = \langle \sum \alpha_i\lambda_i g_i\,, \sum \alpha_i\lambda_i g_i\rangle \underset{\text{orthogonality}}{=} \sum \alpha_i^2\lambda_i^2 \leq \left(\sum \alpha_i^2\right) \cdot \lambda_{max}^2.$$
$$\|f\|^2 = \langle f\,, f\rangle = \sum \alpha_i^2.$$

□

An expander graph can be viewed as a linear operator that approximates J in the operator norm.

Claim 2.3. *If A is the transition matrix of a γ-spectral-expander then $\|A - J\| \leq \gamma$. In particular, for all f $\|(A - J)f\| \leq \gamma \cdot \|f\|$*

Proof. Let f_i be the eigenfunction of A corresponding the the ith eigenvalue λ_i. Clearly $f_1 = \mathbb{1}$. Notice $A\mathbb{1} = \mathbb{1}$ and $J\mathbb{1} = \mathbb{1}$ thus $(A-J)\mathbb{1} = 0$. Moreover, for $i > 1$, $f_i \perp \mathbb{1} : \|Af_i\| \leq \gamma f_i$ and $Jf_i = 0$, therefore $(A-J)f_i = Af_i - 0 = \lambda_i f_i$. We get

$$(A - J)f_i = \begin{cases} 0 & i = 1 \\ \lambda_i f_i & i \geq 1 \end{cases}$$
$$\Rightarrow \|A - J\| = \lambda_{max}(A - J) = max\{|\lambda_2(A)|, |\lambda_n(A)|\} \leq \gamma$$

□

2.1 Random walks on graphs

A random walk on a graph $G = (V, E)$ is a sequence (either finite or infinite), of vertices from V (e.g $(v_0, v_1, v_2, v_1, ...)$). There are many types of random walks with different properties. One such type of random walk can be described by the following algorithm:

Definition 2.4 (Up-down random walk)**.** *On graph G*

- *Start at a random vertex $v_0 \sim \pi$*
  Suppose we are at some vector v_t. Choose v_{t+1} by the following process:
- ***UP:** Choose a random edge e containing v_t*
- ***DOWN:** Choose a random vertex $u \in e$ (uniformly): $v_{i+1} = u$*

Notice: there is always probability $1/2$ to stay in place, which makes the transition matrix of the up-down random walk to be

$$T_{\text{UP-DOWN}} = \frac{1}{2}Id + \frac{1}{2}A$$

where A is the transition matrix of the non-lazy up-down random walk (where in the down step we chose $v_{t+1} \neq v_t$).

To ease our understanding of random walks, consider a bipartite graph $B(G)$ with two parts V_0, V_1 such that $V_0 = V_G, V_1 = E_G$. i.e. a bipartite graph that is made of a set of vertices V_0 which represents **vertices** in G and a set of vertices V_1 which represents **edges** in G. In the bipartite graph $B(G)$ there will be edges between vertices $v \in V_0, e \in V_1$ if $v \in e$. i.e. $E_M = \{(v, e)$ if $v \in e$ for $v \in V_G, e \in E_G\}$. Notice that the the vertices in V_1 will be 2-regular.

Figure 1: Example of a graph G and its appropriate graph $B(G)$

(a) Graph G

(b) $B(G)$

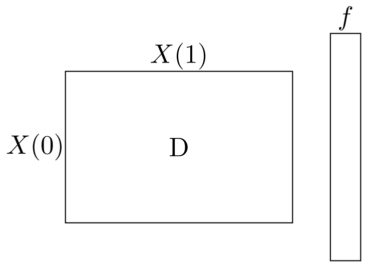
Looking more carefully, we can break the random walk into two steps that are being executed alternately over and over again. These are the selection of edge (up) and the selection of a vertex in the edge (down). Which leads us to define and examine the following operators:

Definition 2.5. *The Down operator* $D : \ell_2(X(1)) \to \ell_2(X(0))$ *is defined by*

$$Df(v) \triangleq \mathbb{E}_{e|v} f(e)$$

(recall our notation $e|v$ is an edge e which contains v, $v|e$ is a vertex v contained in e). More generally $s'|s$ is a face s' that is either containing s or contained in s.

The Down operator is a linear operator, it may be helpful to think of the matrix form:



Notice that $D[v,e] = Pr[e|v]$ and that D is row-stochastic, i.e. $\sum_{e \in E} D[v,e] = 1$.

Definition 2.6. *The Up operator* $U : \ell_2(X(0)) \to \ell_2(X(1))$ *is defined by*

$$Ug(e) = \mathbb{E}_{v|e} g(v) = \mathbb{E}_{v|(v_1,v_2)} g(v) = \frac{1}{2} g(v_1) + \frac{1}{2} g(v_2)$$

as there are exactly two (uniform) choices to choose a vertex from an edge.

The up-down random walk we have defined in Definition 2.4 is actually a composition of operators:

$$\ell_2(V) \xrightarrow{U} \ell_2(E) \xrightarrow{D} \ell_2(V)$$
$$DU : \ell_2(V) \to \ell_2(V)$$
$$DUg(v) = \mathop{\mathbb{E}}_{v' \sim v} G(V) = \mathop{\mathbb{E}}_{\substack{e|v \\ v'|e}} G(V) = \frac{1}{2} Id + \frac{1}{2} A \tag{2.1}$$

Where A is the adjacency matrix of the graph. One may think of this as a two step walk on the bipartite graph $B(G)$ starting from the v side.

2.2 Random walks in higher dimension

Let us now introduce the higher dimensional version of the up-down random walk. From now on, this will be referred to as the upper random walk.

Definition 2.7 (Upper random walk)**.** *For* $k \le d - 1$

- *Start at a random k-face* $s \sim \pi_k$
  Suppose at time t we are at some face s_t*. Choose* s_{t+1} *by the following process:*
- *UP: Choose* s' *by choosing uniformly random* $(k+1)$*-face that contains* s_t*.*
- *DOWN: Choose* s_{t+1} *to be a k-face by choosing uniformly random a single vertex from face* s' *and erase it* ($s_{t+1} \subset s'$)*.*

Let us now introduce the higher dimensional version for the up and down operators:

- **Down operator** $D : \ell_2(X(i+1)) \to \ell_2(X(i))$ is defined by $Df(s) \triangleq \mathbb{E}_{s'|s} f(s')$ where $s \subset s'$ i.e. the expectation is conditioned on s' that **contains** s.
- **Up operator** $U : \ell_2(X(i)) \to \ell_2(X(i+1))$ is defined exactly the same but for $s' \subset s$ by $Uf(s) \triangleq \mathbb{E}_{s'|s} f(s')$ i.e. the expectation is conditioned on s' that **is a subset** of s.

Having the down operator generalized for higher dimensions, we can also introduce another type of random walk which is defined as applying the down operator before the up operator.

Definition 2.8 (Lower random walk)**.** *for $d \geq k \geq 0$*

- *Start at random k face $s_0 \sim \pi_k$*
  Suppose at time t we are at some face s_t. Choose s_{t+1} by the following process:
- *DOWN: Choose s' to be a $(k-1)$-face by choosing uniformly random single vertex from face s_t and remove it.*
- *UP: Choose s_{t+1} by choosing a random (k)-face that contains s_t ($s' \subset s_{t+1}$, according to $\pi_k|s_t$)*

Notice: Performing the lower random walk with $i = 0$ is going down to $X(-1) = \{\emptyset\}$. Performing the up step from the empty set is equivalent to choosing some vertex $v \sim \pi_0$, i.e. selecting a vertex distributed according to the stationary distribution. This means that v_{t+1} is chosen independently from v_t. This is nothing but J_π.

We have just defined the upper random walk and the lower random walk for any simplicial complex.

How do the upper random walk and lower random walk relate to each other?

Example: Let us compute the random walk of DU and UD on a 1-dimensional complex (i.e. a graph). **Reminder:**

$$UDf = \begin{bmatrix} - & \pi_0 & - \\ - & \vdots & - \end{bmatrix} f \tag{2.2}$$

$$DUf = \left(\frac{1}{2}Id + \frac{1}{2}A\right) \cdot f \tag{2.3}$$

where A is the transition matrix of the graph. Notice that for the upper random walk (DUf) there is a $1/2$ probability to stay at the same vertex (starting at vertex v, choosing an edge, and then a vertex in the edge, has probability $1/2$ to choose v). Such a behavior is called $1/2$-lazy random walk. We've seen Equation 2.3 as a result of Equation 2.1 and the analysis that followed.

Definition 2.9 (Non-lazy upper random walk)**.** *The non-lazy upper random walk has 0 probability to return to same face/vertex, i.e. for $k \leq d-1$*

- *Start at a random k-face $s \sim \pi_k$*
  Suppose at time t we are at some face s_t. Choose s_{t+1} by the following process:
- *UP: Choose s' by choosing a random $(k+1)$-face that contains s_t according to $\Pi_{k+1}|s_t$*
- *DOWN: Choose s_{t+1} to be a k-face by choosing uniformly at random a single vertex from face s' and erase it ($s_{t+1} \subset s'$)* ***s.t.*** $\mathbf{s_{t+1} \neq s_t}$

i.e.

$$A = 2DU - ID$$

Where A is the relevant adjacency matrix of the graph. This derives directly from Equation 2.3.

Describing the lazy upper random walk on faces in $X(k)$ we get that $DU = \frac{k+1}{k+2}A_k + \frac{1}{k+2}Id$ where A_k is a transition matrix between faces in $X(k)$. From this formulation we can derive algebraic description for the non-lazy upper random walk $A_k = \frac{k+2}{k+1}DU - \frac{1}{k+1}Id$

Let M_k^+ denote the non-lazy upper random walk transition operator (A_k from above). If k is clear from the context, we omit it and write M^+.

2.3 Random walks on spectral γ-bounded high dimensional expander

We have seen in Claim 2.3 that in expander graphs

$$\| \underset{\underset{UD(\text{lower})}{||}}{J_\pi} - \underset{\underset{M^+}{||}}{A} \| \leq \gamma.$$

We will now describe random walks on complexes to define expansion of complexes.

Notations:

- Down operator from $X(k+1)$ to $X(k)$: $D_{\searrow k} = D_{k+1\searrow}$
- Up operator to $X(k)$ from $X(k-1)$: $U_{\nearrow k} = U_{k-1\nearrow}$
- M_k^+ non-lazy upper random walk moving from $X(k)$ to $X(k)$

(When the context is clear we may omit the index k)

Claim 2.10. $\langle U_{\nearrow k+1} g \, , f \rangle_{X(k+1)} = \langle g \, , D_{\searrow k} \rangle_{X(k)}$

Proof.

$$\begin{aligned}
\langle U_{\nearrow k+1} g \, , f \rangle_{X(k+1)} &= \underset{s\in\pi_{k+1}}{\mathbb{E}} [U_{\nearrow k+1} g(s) \cdot f(s)] \\
&= \underset{s}{\mathbb{E}} \underset{s'|s}{\mathbb{E}} \, g(s') \cdot f(s) \\
&= \underset{s'}{\mathbb{E}} \underset{s|s'}{\mathbb{E}} \, f(s) \cdot g(s') \\
&= \underset{s'}{\mathbb{E}} Df(s') \cdot g(s') \\
&= \langle g \, , D_{\searrow k} f \rangle_{X(k)}
\end{aligned}$$

□

Definition 2.11 (γ random walk high dimensional expander)**.** *A d-dimensional complex X is a γ random walk high dimensional expander if for every k s.t.* $0 \leq k \leq d-1$

$$\left\| M_k^+ - U_{k\nearrow} D_{k\searrow} \right\| \leq \gamma$$

i.e. a complex is γ-random-walk high dimensional expander if the distance between the non-lazy upper random walk and the lower random walk is bounded by γ for every dimension $0 \leq k \leq d-1$.

Lemma 2.12. *If X is a d-dimension γ-two-sided-link expander then it is a γ-random-walk expander*

Recall: X_s is the link of s where s is some face in X .

Recall: X is a γ-link expander if for every $s \in X(k), -1 \leq k \leq d-2 :\quad \lambda_{max}(X_s) \leq \gamma$ (in this notation we refer to the maximal absolute eigenvalue ($\lambda_{max} = max(|\lambda_2|, |\lambda_n|)$) of the 1-skeleton of X_s).

Proof. We will prove the lemma for $d = 2$

for $k = 0$:

$$\text{We need to show: } \|M_0^+ - U_{\nearrow 0} D_{0\searrow}\| \leq \gamma$$
$$\|A - J_\pi\| \leq \gamma \text{ (By definition of expansion of the link of } \emptyset)$$

This follows since M_0^+ is the non lazy upper random walk, (i.e. $M_0^+ = 0 \cdot Id + 1 \cdot A = A$) and $U_{\nearrow 0} D_{0\searrow}$ is the lower random walk to $X(-1)$ which the operator J_π introduced previously

for $k = 1$: We need to show $\left\|M_1^+ - U_{\nearrow 1} D_{1\searrow}\right\| \leq \gamma$

It is enough to show $|\langle (M_1^+ - UD)f\,, f\rangle| \leq \gamma \langle f\,, f\rangle$.

For each $v \in X(0)$ we define $f^v : X_v(0) \to \mathbb{R}$ as a localization of f to vertices in the link of v. i.e. for $u \in X_v(u) \quad f_v(u) := f((v,u))$.

$$\begin{aligned}
\langle M_1^+ f\,, f\rangle &= \mathop{\mathbb{E}}_{e} M_1^+ f(e) \cdot f(e) = \mathop{\mathbb{E}}_{e \sim e'} f(e) \cdot f(e') \\
&= \mathop{\mathbb{E}}_{\substack{t \in X(2) \\ \text{triangle}}} \mathop{\mathbb{E}}_{e_1 \neq e_2 \subset t} f(e_1) \cdot f(e_2) \\
&= \mathop{\mathbb{E}}_{v} \mathop{\mathbb{E}}_{\substack{v \in t = \{u,v,w\} \\ e_1 = \{v,u\} \\ e_2 = \{v,w\}}} f(e_1) \cdot f(e_2) \\
&= \mathop{\mathbb{E}}_{v} \langle A_v f^v\,, f^v\rangle
\end{aligned}$$

$$\begin{aligned}
\langle U_{0\nearrow} D_{\searrow 0} f\,, f\rangle &= \langle D_{\searrow 0} f\,, D_{\searrow 0} f\rangle \\
&= \mathop{\mathbb{E}}_{v} D_{\searrow 0} f(v) \cdot D_{\searrow 0} f(v) \\
&= \mathop{\mathbb{E}}_{v} \left[\mathop{\mathbb{E}}_{e_1 | v} f(e_1)\right] \left[\mathop{\mathbb{E}}_{e_2 | v} f(e_2)\right] \\
&= \mathop{\mathbb{E}}_{v} \mathop{\mathbb{E}}_{\{u,v\},\{w,v\} | v} f^v(\{u,v\} \setminus v) f^v(\{w,v\} \setminus v) \\
&= \mathop{\mathbb{E}}_{v} \mathop{\mathbb{E}}_{u,w \sim X_v(0)} f^v(u) f^v(w)
\end{aligned}$$

To summarize, let f be an eigenfunction of eigenvalue λ_i for $i \geq 2$ s.t. $\|f\| = 1$. As such $f \perp \mathbb{1}$ and so $Jf = 0$ (As $Jf = 0$ for all $f \perp \mathbb{1}$).

$$\begin{aligned}
|\langle M^+ f\,, f\rangle - \langle UDf\,, f\rangle| &= |\mathop{\mathbb{E}}_{v} \mathop{\mathbb{E}}_{(u,w) \sim x_v(1)} f^v(u) f^v(w) - \mathop{\mathbb{E}}_{v} \mathop{\mathbb{E}}_{u,w \sim X_v(0)} f^v(u) f^v(w)| \\
&= \mathop{\mathbb{E}}_{v} |\mathop{\mathbb{E}}_{(u,w) \sim x_v(1)} f^v(u) f^v(w) - \mathop{\mathbb{E}}_{u,w \sim X(0)} f^v(u) f^v(w)| \\
&= \mathop{\mathbb{E}}_{v} |\langle A_v f^v\,, f^v\rangle - \langle J_v f^v\,, f^v\rangle| \\
&= \mathop{\mathbb{E}}_{v} |\langle (A_v - J_v) f^v\,, f^v\rangle| \\
&\leq \mathop{\mathbb{E}}_{v} |\gamma \left\|f^v\right\|^2| = \gamma \mathop{\mathbb{E}}_{v} \left\|f^v\right\|^2
\end{aligned}$$

The last inequality follows Claim 2.3 because the link of v is an expander graph, as X is a γ-link-expander. $\square$