High Dimensional Expanders

Lecture 4: The Flag Complex

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In this lecture we construct a d-dimensional simplicial complex (definition 2.1), named the spherical building or the flags complex. We give an elementary description of this complex and investigate the link structure, computing their spectral properties using the general theory of expansion for bipartite graphs (subsection 1) and relying on Garlands method via Oppenheim's theorem (subsection 2.2).

1 Preliminaries: expansion of bipartite graphs

Let $G=(V,E)$ be a connected bipartite graphs. Let $V=A\sqcup B$ then as $Pr\left[E(A,A)\right]=0$ and $Pr\left[E\left(B,B\right)\right]=0$ the adjacency matrix takes the form $M=\begin{bmatrix}0 & T\\ T^{*} & 0\end{bmatrix}$, where the zero blocks correspond to the probabilities to traverse from A to A or from B to B. We shall now find relation between the eigenvalues of $T^{*}T,TT^{*}$ and M.

Claim 1.1. *If λ is an eigenvalue for M then λ^{2} is an eigenvalue for $T^{*}T$ and TT^{*}*

Proof. Let $f\in l_{2}\left(V\right)$ such that $Mf=\lambda f$. Set $f_{A}:A\to\mathbb{R}$ be $f_{A}\left(a\right)=f\left(a\right)$ and similary set f_{B}. Then $Tf_{B}=\lambda f_{A}$ and $T^{*}f_{A}=\lambda f_{B}$, thus $T^{*}Tf_{B}=T^{*}\left(\lambda f_{A}\right)=\lambda^{2}f_{B}$ and λ^{2} is an eigenvalue of $T^{*}T$ similary λ^{2} is an eigenvalue of TT^{*}. On the other hand □

Claim 1.2. *If $\lambda\neq0$ is an eigenvalue of $T^{*}T$ then $\pm\sqrt{\lambda}$ are eigenvalues of M.*

Proof. If $T^{*}Tg=\lambda g$, then as $T^{*}T$ is positive definte $\lambda>0$. Let $f=Tg$ and define

$$h\left(v\right)=\begin{cases}\frac{f(v)}{\sqrt{\lambda}} & v\in A\\ g\left(v\right) & v\in B\end{cases},$$

then

$$Mh=\begin{cases}Tg\left(v\right) & v\in A\\ T^{*}\frac{f(v)}{\sqrt{\lambda}} & v\in B\end{cases}=\begin{cases}f\left(v\right) & v\in A\\ \frac{T^{*}Tg(v)}{\sqrt{\lambda}} & v\in B\end{cases}=\begin{cases}f\left(v\right) & v\in A\\ \sqrt{\lambda}g\left(v\right) & v\in B\end{cases}=\sqrt{\lambda}h,$$

So $\sqrt{\lambda}$ is an eigenvalue of M. Note that similarly $-\sqrt{\lambda}$ is also an eigenvalue of M. □

Remark 1.3. *There is a similar result for TT^{*}.*

2 The Flag Complex

Let $q=p^{n}$ be a power of some prime p. Consider the field $\mathbb{F}_{q}$ and the vector space $\mathbb{F}_{q}^{m}$, a line in $\mathbb{F}_{q}^{m}$ is a subspace of the form $l=\operatorname{span}_{\mathbb{F}_{q}}\left(\{a\}\right)$ for some $0\neq a\in\mathbb{F}_{q}^{m}$.

Let us denote $Gr\left(m,k\right)$ as the set of all k-dimensional subspaces of $\mathbb{F}_{q}^{m}$. [1]

Definition 2.1. *The flag complex of dimension $d=m-2$ is given by:*

$$X\left(0\right)=\bigcup_{k=1}^{m-1}Gr\left(m,k\right),$$

[1]Note that $\left|Gr\left(m,1\right)\right|=\frac{q^{m}-1}{q-1}$.

$$X(1)=\{\{V,W\}\,|V\subset W \quad V,W\in X(0)\}$$

and X is a clique complex, i.e $\{A_1,...,A_{i+1}\}$ is a clique in $(X(0),X(1))$ if and only if $\{A_1,...,A_{i+1}\}\in X(i)$. X has the uniform distribution on $X(d)$ and the induced distributions for $i<d$.

Definition 2.2. *A sequence $0\subsetneq V_1\subsetneq V_2\subsetneq ...\subsetneq V_{m-1}\subsetneq \mathbb{F}_q^m$ is called a flag.*

Claim 2.3. *Every d-face of the flag complex is a flag.*

Proof. By induction on d, for $d=1$ a d-face is just an edge, $\{A_1,A_2\}$ so $0\subsetneq A_1\subsetneq A_2\subsetneq \mathbb{F}_q^m$ is a flag. Next let $\{A_1,...,A_{m-1}\}$ be a clique in $(X(0),X(1))$ and WLOG assume that A_1 is of smallest dimension. As $\{A_1,...,A_{m-1}\}$ is a clique and A_1 is of smallest dimension we must have that $A_1\subsetneq A_i$ for all $i\geq 2$. Let $Q_i=A_i/A_1$ then Q_i is a $d-1$ in the flag complex of dimension $d-1$ thus by the induction hypothesis a flag

$$0\subsetneq Q_2\subsetneq ...\subsetneq Q_{m-1}\subsetneq \mathbb{F}_q^m/A_1=\mathbb{F}_q^{m-1}$$

but then

$$0\subsetneq A_1\subsetneq A_2\subsetneq ...\subsetneq A_{m-1}\subsetneq \mathbb{F}_q^m$$

is also a flag.

Let us consider the case $d=1$. In this case our vertices are lines and planes in $\mathbb{F}_q^3$. The number of lines is q^2+q+1, and by the dualty of lines and planes [2] we have that the number of planes is also q^2+q+1. The flag complex we get is a bipartite graph. Let us find the degree of a plane, for a fixed plane P the number of lines contained in P is:

$$\frac{\#\text{of non-zero points in the plane}}{\#\text{of non-zero points that span the same line}}=\frac{q^2-1}{q-1}=q+1.$$

And the degree of the line L is given by the number of planes containing it, which is:

$$\frac{\#\text{of points outside L}}{\#\text{of points outside } L \text{ spanning the same plane}}=\frac{q^3-q}{q^2-q}=q+1$$

We shall now disscus spectral properties of more general bipartite graphs so we would have the tools to compute the spectral properties of the flag grpah and links in the flag complex.

2.1 The one dimensional flag complex (graph)

Let us focus on the one dimensional flag complex. This is a bipartite graph whose vertices are all 1-dimensional (lines) and 2-dimensional (planes) linear spaces in a 3 dimensional vector space $\mathbb{F}_q^3$. An edge connects a line to a plane if the line is in the plane. If we take a length two path in this graph originating at a line, we have a $\frac{1}{q+1}$ chance to return to the original line, and equal probability to reach all other lines (as any two different lines in $\mathbb{F}_q^3$ span a plane) hence,

$$T^*T=\frac{1}{q+1}Id+\left(1-\frac{1}{q+1}\right)K,\text{ denote }\alpha=\frac{1}{q+1}.$$

Let J be the matrix with all entries $\beta=\frac{1}{q^2+q+1}$ then $J=\beta Id+(1-\beta)K$ and by a simple computation

$$T^*T=\frac{1-\alpha}{1-\beta}J+\frac{\alpha-\beta}{1-\beta}Id,$$

[2]In 3 dimensions a planes is determined by a single equation $ax+by+cz=0$ where a,b,c are the coordinates of the dual line.

so the eigenvalues of T^*T are 1 and

$$\frac{\alpha-\beta}{1-\beta}=\frac{1}{q+1}\frac{q^2+q+1-(q+1)}{q^2+q+1-1}=\frac{1}{q+1}\frac{q^2}{q^2+q}<\frac{1}{q+1}$$

Hence the eigenvalues of M are ± 1 and roughly $\pm\frac{1}{\sqrt{q+1}}$, so the one dimensional flag complex is one sided expander with $\lambda<\frac{1}{\sqrt{q+1}}$.

2.2 Links in the flag complex

We now return to the d-dimensional flag complex, and explore the structure of the links and their expansion. Given $\sigma\in X(d-2)$, one can always write

$$\sigma=\{V_1\subset V_2\subset\cdots\subset V_{i-1}\subset V_{i+1}\subset\cdots\subset V_{j-1}\subset V_{j+1}\subset\cdots\subset V_{m-1}\}$$

for a pair of indices $1\le i<j\le m-1$.

There are several cases:

- Suppose first that $i=1$ and $j=2$. In this case X_σ is isomorphic to the one dimensional flag complex (whose vertices are 1-dimensional and 2-dimensional spaces in V_3), thus $\lambda(X_\sigma)\le\frac{1}{\sqrt{q+1}}$.
- If $j=i+1$ then σ has is an increasing sequence of subspaces of all dimensions except $i,i+1$. A vertex in this link is a subspace V such that

  $$A_{i-1}\subset V\subset A_{i+2}.$$

  Every $V\supset A_{i-1}$ is determined by V/A_{i-1} so it is convenient to pass to the quotient space (looking at cosets of A_{i-1}),

  $$\{0\}=A_{i-1}/A_{i-1}\subset V/A_{i-1}\subset A_{i+2}/A_{i-1}\backsimeq\mathbb{F}^3$$

  The combinatorial incidence structure becomes the same as the case of $i=1,j=2$. We get that X_σ is the one dimensional flag complex, thus $\lambda(X_\sigma)<\frac{1}{\sqrt{q+1}}$.
- If σ has subspaces of all dimensions except i,j such that $i>j+1$ then X_σ actually has a simpler structure. One can check that it is the complete bipartite graph with $q+1$ vertices on each side. Indeed fixing any two linear spaces U_1,U_2 such that $\dim U_2-\dim U_1=2$, the number of subspaces V sandwiched between them $U_1\subset V\subset U_2$ is exactly $q+1$. Clearly this is an expander and in particular $\lambda(X_\sigma)<\frac{1}{\sqrt{q+1}}$.

So we get a uniform bound on all links of dimension 1, $\lambda_{d-2}<\frac{1}{\sqrt{q+1}}$. For $\sigma\in X(d-3)$ we can use Oppenheim's theorem (from Lecture 1) to get $\lambda(X_\sigma)\le\frac{\lambda_{d-2}}{1-\lambda_{d-2}}=\frac{1}{\sqrt{q+1}-1}$, continuing inductivly if $\dim(\sigma)=d-k$ then $\lambda(X_\sigma)\le\frac{1}{\sqrt{q+1}-k+2}$, thus for all $\sqrt{q}\gg d$ we get expansion all the way. $\square$