High Dimensional Expanders

Lecture 7: Coboundry Expanders

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In this lecture we present **coboundry expansion**. We will first introduce the edge expansion, sometimes called the Cheeger Constant which is a special case of coboundary expansion. Afterward, we will present key terms related to Homology & Cohomology. We then introduce Coboundary Expansion by describing coboundary of a graph. Finally, we will present briefly several Context & Applications for research of coboundary expansion.

1 Conductance

Definition 1.1 (Cheeger Constant). *Let $G = (V, E)$ a d-regular graph, and $S \subset V$ s.t. $0 < |S| \leq \frac{|V|}{2}$. Denote $E(S, \overline{S}) = \{e \in E || e \cap S| = 1\}$, then the **Conductance of** G is defined as*

$$H_G = \min_S \frac{\frac{|E(S,\overline{S})|}{|E|}}{\frac{|S|}{|V|}}$$

Theorem 1.2 (Cheeger-Alon-Milman-Dodziuk). *Let G be a graph and let λ_2 the second largest eigenvalue of the normalized adjacency operator of G*

$$\frac{1-\lambda_2}{2} \leq H_G \qquad \text{"Easier" side by Alon-Milman}$$

$$H_G \leq \sqrt{2(1-\lambda_2)} \qquad \text{"Harder" side by Alon, Dodziuk}$$

Combining both, we get that

$$\frac{1-\lambda_2}{2} \leq H_G \leq \sqrt{2(1-\lambda_2)}.$$

Our interpretation is that a lower bound on the spectral gap (equivalently: an upper bound on λ_2) means a lower bound on the conductance of G. The best approximation algorithm known for conductance gives $\sqrt{\log(n)}$ factor, due to Arora-Rao-Vzirani [**ARV09**].

Open Question: Is there a conductance approximation algorithm with a constant factor (i.e. better than $\sqrt{log(n)}$)?

2 Homology & Cohomology

So far we looked at specific functions from faces of $X(i)$ to the reals, and in particular at $f : V \to \mathbb{R}$. We will now look at functions from $X(i)$ to $\mathbb{F}_2 = \{0, 1\}$. More precisely, let i-cochain be

$$C_i = \{f : X(i) \to \{0, 1\}\} \qquad \text{“Cochains”}$$

Where the cochain C_{-1} is the set of the two constant functions from the empty set to either 0 or 1, i.e. $C_{-1} = \{f : \emptyset \to \{0, 1\}\} \cong \{0, 1\}$. Notice that every function $f \in C_i$ can be looked at as representing a subset of faces in $X(i)$. It can be seen by considering a set $S \subseteq X(i)$ that is defined by $\{s \in X(i)|f(s) = 1\}$. We will consider the following operators:

Definition 2.1 (Boundary and Coboundary operators)**.**

Boundary

$\partial_i : C_i \to C_{i-1}$

$\forall S \in X(i-1) \quad \partial_i f(S) := \sum_{\substack{T \supset S \\ T \in X(i)}} f(T)$

Coboundary

$\delta_i : C_i \to C_{i+1}$

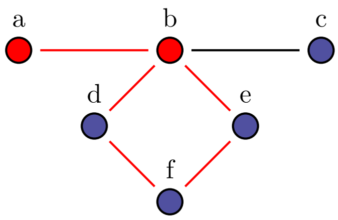
$\forall A \in X(i+1) \quad \delta_i f(A) := \sum_{\substack{T \subset A \\ T \in X(i)}} f(T)$

(All summation is in the field $\mathbb{F}_2$).

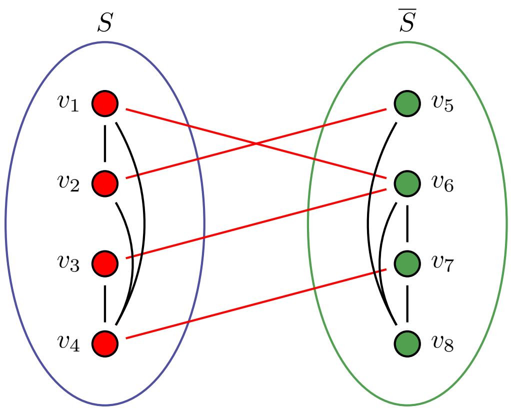
Examples:

$$f : X(1) \to \{0,1\}$$
$$\partial f : X(0) \to \{0,1\}$$
$$\delta f : X(2) \to \{0,1\}$$

Figure 1: Boundary of a function on edges



(a) $f(e) = 1$ on red edges, 0 otherwise. $\partial f(v) = 1$ on red vertices where the sum (in $\mathbb{Z}_2$) equals 1, 0 otherwise



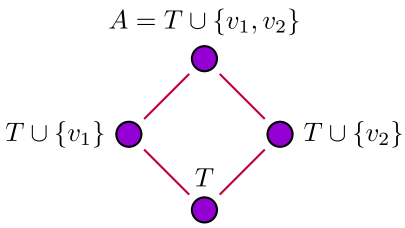
(b) Let $S \subseteq X(0)$ and $g(v) = 1$ for $v \in S$, 0 otherwise. $\delta g(e) = 1$ for edges in $E(S, \overline{S})$. For any other edge $e = (u, v)$, $g(u) + g(v) = 0$.

Lemma 2.2. *$\delta_{i+1} \circ \delta_i = 0$ and similarly $\partial_i \circ \partial_{i+1} = 0$*

Proof. Enough to show that for every face $A \in X(i+1)$ $\partial\partial 1_A = 0$, where 1_A is indicator of A. Let $A \in X(i+1)$ and 1_A as described. Then for every $T \in X(i-1)$ one of the two follows:

- $T \not\subset A$: In that case $\partial\partial 1_A(T) = 0$ by definition.
- $T \subset A$: Then there exists two vertices v_1, v_2 such that $T \cup \{v_1, v_2\} = A$, and we get that

$$\partial 1_A(A \setminus \{v_1\}) = \partial 1_A(T \cup \{v_1\}) = 1$$
$$\partial 1_A(A \setminus \{v_2\}) = \partial 1_A(T \cup \{v_2\}) = 1$$



Which implies that

$$\begin{aligned}
\partial\partial 1_A(T) &= \sum_{v\in X_T(0)} \partial 1_A(T\cup\{v\}) \\
&= \sum_{v\in\{v_1,v_2\}} \partial 1_A(T\cup\{v\}) + \underbrace{\sum_{v\in X_T(0)-\{v_1,v_2\}} \partial 1_A(T\cup\{v\})}_{=0} \\
&= \partial 1_A(T\cup\{v_1\}) + \partial 1_A(T\cup\{v_2\}) \\
&= 1+1=0
\end{aligned}$$

Where the last equality is as all summation is modulo 2.

The proof for $\delta\delta = 0$ is similar.

$\square$

Definition 2.3.

$B_i = \operatorname{Im}\partial_{i+1} \subset C_i$	***Boundary***	$B^i = \operatorname{Im}\delta_{i-1} \subset C_i$	***Coboundary***
$Z_i = \ker\partial_i \subset C_i$	***Cycle***	$Z^i = \ker\delta_i \subset C_i$	***Cocycle***
$H_i = Z_i/B_i$	***Homology***	$H^i = Z^i/B^i$	***Cohomology***

Corollary 2.4. *It follows from $\delta_{i+1}\circ\delta_i = 0$, that $\operatorname{Im}\delta_{i+1} \subseteq \ker\delta_i$ i.e. $B^i \subseteq Z^i$. Similarly $\partial_i\circ\partial_{i+1} = 0$, implies that $\operatorname{Im}\partial_{i+1} \subseteq \ker\partial_i$ that is $B_i \subseteq Z_i$. Therefor it makes sense to define H_i and H^i.*

Example: Cohomology of graph

Let $G = (V,E)$, let's calculate $H^0 = Z^0/B^0$. Notice that C_{-1} is the family of functions from the empty set to $\{0,1\}$. Therefore, the coboundary B^0 is the set of constant functions on the vertices, i.e. $B^0 = \operatorname{Im}\delta_{-1} = \{\mathbb{1},\mathbb{0}\}$. Let $f \in Z^0$ i.e. $f : V \to \mathbb{Z}_2$ and $\delta_0 f$ is the constant zero function on edges. i.e. for every edge $(u,v) = e \in E$ it holds that $\delta_0 f(e) = f(u)+f(v) = 0 \Leftrightarrow f(u) = f(v)$ for every $u,v \in V$. We can conclude that $f \in Z^0$ is constant on every connected component in V. **Conclusion**: $Z^0 = B^0$ iff G is connected. In this case $H^0 = \{0\}$ and $\dim H^0 = 0$.

Definition 2.5. *X is k-chomologically-connected if for every $i \le k$ it holds that $Z^i = B^i$ ($H^i = \{0\}$).*

3 Coboundary Expansion

We saw that $H^0 = \{0\}$ iff the graph is connected. We now study a more quantitative measure to connectivity, called coboundary expansion. Let $f : V \to \{0,1\}$ and denote $S = \{v \in V \text{ s.t. } f(v) = 1\}$ and equivalently for $f(v) = 0$ denote $\overline{S} = \{v \in V \text{ s.t. } f(v) = 0\}$. As discussed, $\delta_0 f(e) = 1$ if and only if $|e\cap S| = 1$, i.e. $\delta_0 f$ is the indicator of the edges of the cut $(S,\overline{S})$. For any function $h \in C_1$ define

$$\operatorname{wt}(h) = \sum_{\sigma:h(\sigma)=1} \operatorname{wt}(\sigma).$$

This is equivalent to the expectation of $h(\sigma)$ when choosing σ ar random according to Π_i. For the uniform distribution over $X(1)$ this is

$$\operatorname{wt}(h) = \frac{|\{\sigma \in X(1) \text{ s.t. } h(\sigma) = 1|}{|X(1)|}.$$

Let us now look at the ratio between the weight of a function to the weight of its coboundary

$$\frac{\mathrm{wt}(\delta_0 f)}{\mathrm{wt}(f)} = \frac{\frac{E(S,\overline{S})}{|E|}}{\frac{|S|}{|V|}}.$$

This is exactly the conductance of G as per Definition 1.1.

Recall: The Cheeger Constant:

$$H_g = \min_{\substack{S\subset V \\ 0\le|S|\le|V|/2}} \frac{\frac{|E(S,\overline{S})}{|E|}}{\frac{|S|}{|V|}} = \min_{\emptyset\neq S\neq V} \frac{\frac{|E(S,\overline{S})|}{|E|}}{\frac{\min(|S|,|\overline{S}|)}{|V|}}.$$

Notice: $\min(|S|,|\overline{S}|)$ is the hamming distance of the indicator function for S (equivalently $\overline{S}$) from the all 1s all 0s function. i.e. the distance of S from V and $\emptyset$.

Notation: Let f, f' be two functions such that $f, f' : D \to R$. We denote $\mathrm{dist}(f, f')$ as the fraction of disagreeing inputs between f and f'. And for a set A of functions from D to R, denote $\mathrm{dist}(f, A) = \min_{f'\in A} \mathrm{dist}(f, f')$.

Definition 3.1 (Coboundary expansion)**.**

$$h_k = \min_{f\in C_k\setminus B^k} \frac{\mathrm{wt}(\delta_k f)}{\mathrm{dist}(f, B^k)}.$$

Notice that the coboundary expansion for $k = 0$ i.e. h_0 is in fact the normalized Cheeger constant. This comes from the fact that B^0 consists of the all 1s and all 0s functions. This can be easily seen by moving from any function f to the set $S = \{v : f(v) = 1\}$.

$$h_0 = \min_{f\notin B^0} \frac{\mathrm{wt}(\delta_0 f)}{\mathrm{dist}(f, B^0)} = \min_{f\neq 1,0} \frac{\frac{|E(S,\overline{S})|}{|E|}}{\frac{\min(|S|,|\overline{S}|)}{|V|}}$$

Remark 3.2. *If $h_k = 0$ then there exists a function $f \in C_k - B^k$ s.t. $\delta_k f = 0$ i.e. $f \in Z^k$. f is also not in B^k, so we get $f \in Z^k/B^k = H^k$ so we can conclude that $H^k \neq \emptyset$. To summarize $h_k = 0 \Leftrightarrow H^k \neq 0$.*

4 Context & Applications

1. Linial and Meshulam [**LM03**] wanted to learn expansion properties of random complexes, this is a generalization of $G(n, p)$ which is a graph on n vertices with independent probability p for each edge's existence. They found that connectivity of random complex happens in probability $\frac{k\cdot log(n)}{n}$, this result comes from understanding the coboundary expansion of the complete complex.

2. Gromov [**G10**] introduced the topological overlap property.

   For start, let's consider a different notion of conductance. The geometric idea behind the following definition is that for every drawing of vertices on a line, and edges as intervals between vertices then at least an α fraction of the edges (intervals) will overlap.

   Generalizing this idea to higher dimensions yield the topological overlap definition

   Definition 4.1. *We say that a point $x_0 \in \mathbb{R}^d$* ***pierces*** *a face $S \in X(I)$ in embedding $M : X(0) \underset{mapping}{\hookrightarrow} \mathbb{R}^d$ if x_0 resides is in the convex hull of $M(S)$*

   Example: for an embedding M of $X(0)$ to $\mathbb{R}$, an edge (u, v) induces an interval between $M(u)$ and $M(v)$ and every point in the range $[M(u), M(v)]$ pierces the edge (u, v).

Definition 4.2. *We say that $X^{(d)}$ has the topological overlap property if for every embedding of $X(0) \underset{mapping}{\hookrightarrow} \mathbb{R}^d$ there exists $x_0 \in \mathbb{R}^d$ that pierces c-fraction of $X(d)$.*

Claim 4.3. *If G is a graph with conductance $> \alpha$ then it has the geometric overlap property.*

Proof. Embed vertices on real line and pierce in the median. □

Theorem 4.4. *There exists a point that pierces a constant fraction of topological triangles.*

How: if X is a complex with coboundary expansion $> c$ then X satisfies topological overlap.

3. Kaufman and Lubotzky [**KL13**] relate to property testing and locally testable codes.

   C is a linear locally testable code if $C = \{x \in \{0,1\}^n | Hx = 0\}$ where H is parity check matrix and H is "special" in the sense that every row in H has $\leq q$ entries of 1.

   If $w \in \{0,1\}^n$ s.t. $\mathrm{wt}(Hw) < \epsilon$ then there exists $x \in C$ s.t. $\mathrm{dist}(w, x) < \epsilon$.

   Emphasis: Satisfy all equations

$$\begin{aligned} x_0 + &x_1 + x_2 = 1 \\ x_0 + &x_2 + x_3 = 1 \\ &\vdots \\ x_0 + &x_{100} + x_{101} = 1 \\ x_0 + &x_{101} + x_1 = 1 \end{aligned}$$

   And let's analyze this in term of coboundary expansion. Consider a function $f : X(1) \to \{0,1\}$ i.e. a boolean function on edges. $\delta_1 f : X(2) \to \{0,1\}$ will output 1 for triangles that has an odd number of edges for which f gives 1. Our code would consist of B^1.

References

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[4] Sanjeev Arora, Satish Rao, Umesh Vazirani: Expander Flows, Geometric Embeddings and Graph Partitioning https://www.cs.princeton.edu/~arora/pubs/arvfull.pdf