High Dimensional Expanders

Lecture 9: Coboundary Expansion and Locally Testable Codes

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In this lecture we continue Lecture 7 and study **coboundary expansion**, this time connecting this notion to that of locally testable codes. We will prove that the complete complex is a good coboundary expander and then move to discuss locally testable codes.

1 Recap: Homology, Cohomology, and friends

We recall the definitions from Lecture 7. Let $C_i(X, \mathbb{F})$ denote the vector space of "i-chains", namely functions from $X(i)$ to $\mathbb{F}$. We are focusing on $\mathbb{F} = \mathbb{F}_2$.

Definition 1.1 (Boundary and Coboundary operators)**.**

Boundary

$\partial_i : C_i \to C_{i-1}$

$$\forall S \in X(i-1) \quad \partial_i f(S) := \sum_{\substack{T \supset S \\ T \in X(i)}} f(T)$$

Coboundary

$\delta_i : C_i \to C_{i+1}$

$$\forall A \in X(i+1) \quad \delta_i f(A) := \sum_{\substack{T \subset A \\ T \in X(i)}} f(T)$$

(All summation is in the field $\mathbb{F}_2$).

We defined $B_i = \operatorname{Im} \partial_{i+1}$ and $Z_i = \ker \partial_i$ and also $B^i = \operatorname{Im} \delta_{i-1}$ and $Z^i = \ker \delta_i$. We saw that always $\delta_{i+1} \circ \delta_i = 0$ and $\partial_i \circ \partial_{i+1} = 0$. Therefore it makes sense to define the quotient space called the homology: $H_i = Z_i / B_i$, and the quotient space called the cohomology: $H^i = Z^i / B^i$. (Recall that the elements of a quotient space are cosets, e.g. $z + B^i$ is a typical element in H^i).

Definition 1.2. *X is k-chomologically-connected if for every $i \le k$ it holds that $Z^i = B^i$ (so $H^i = \{0\}$).*

For example, we check that the complete complex is very well connected.

Claim 1.3. *Let X be the complete d-dimensional complex on n vertices. Then for all $i < d$, $Z^i = B^i$.*

Proof. We know that $B^i \subseteq Z^i$ because $\delta_{i-1} \circ \delta_i = 0$, so what we need to show is that $Z^i \subseteq B^i$. Suppose $f : X(i) \to \mathbb{F}$ and assume that $f \in Z^i$, namely $\delta_i f = 0$. We will show that $f \in B^i = \operatorname{Im} \delta_{i-1}$ by showing that $f = \delta g$ for some $g \in C_{i-1}$.

Fix $w_0 \in X(0)$ and set $g(u_1, \ldots, u_i) := f(w_0, u_1, \ldots, u_i)$ whenever $w_0 \notin \{u_1, \ldots, u_i\}$. Otherwise set $g(w_0, u_1, \ldots, u_i) = 0$. Let us check that $f = \delta g$: Suppose that $w_0 \notin \{u_0, \ldots, u_i\}$. We calculate

$$\delta g(u_0, u_1, \ldots, u_i) = \sum_{j=0}^{i} g(u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_i) = \sum_{j=0}^{i} f(w_0, u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_i) = f(u_0, \ldots, u_i)$$

where the last equality follows because $\delta f = 0$ and in particular $\delta f(w_0, u_0, \ldots, u_i) = 0$.

In case that $w_0 \in \{u_0, \ldots, u_i\}$ we get that (suppose $w_0 = u_k$)

$$\delta g(u_0, \ldots, u_i) = \sum_{j \neq k} g(u_0, \ldots, u_{k-1}, w_0, u_{k+1}, \ldots, u_i) + g(u_0, \ldots, u_{k-1}, u_{k+1}, \ldots, u_i) = 0 + f(u_0, \ldots, u_i)$$

where the last equality is because whenever w_0 is inside a face, g is defined to be zero. Whenever w_0 is not inside a face, g is defined to be equal to f on the face when we add w_0. □

2 Coboundary Expansion

We now recall the quantitative definition of connectivity. This definition needs a notion of weight on the $X(i)$. It is simplest to assume first that the complex is regular, namely that every i face is contained in the same number of $i+1$ faces. A more general definition of weights was given in the first lecture:

Definition 2.1. *Let X be a pure d-dimensional simplicial complex. We define on X measures $\pi_d, \pi_{d-1}, \ldots, \pi_0$ as follows:*

- *π_d: an arbitrary probability distribution over $X(d)$, for example the uniform distribution.*
- *π_i: The probability of choosing a face $T' \in X(i)$ is the probability of choosing a face $T \in X(d)$ with the distribution π_d and then choosing T' with uniform distribution over all faces in $X(i)$ that are contained in T.*

With this definition in hand it is natural to define the weight of an i-chain f as

$$\mathrm{wt}(f) := \Pr_{\sigma \sim X(i)}[f(\sigma) \neq 0] = \sum_{\sigma : f(\sigma) \neq 0} wt(\sigma)$$

where summation here is over $\mathbb{R}$. Similarly, the distance of an i-chain to a set B^i is

$$\mathrm{dist}(f, B^i) := \min_{b \in B^i} \mathrm{wt}(f - b).$$

The coboundary expansion compares the distance of f from Z^i (measured by the weight of $\delta_i f$) to the distance of f to B^i.

Definition 2.2 (Coboundary expansion)**.**

$$h^i = \min_{f \in C_i \setminus B^i} \frac{\mathrm{wt}(\delta_i f)}{\mathrm{dist}(f, B^i)}.$$

(We have seen in Lecture 7 that the coboundary expansion for $i = 0$ is in fact the normalized Cheeger constant.)

3 Coboundary Expansion of the Complete Complex

Theorem 3.1 (Gromov, Linial-Meshulam)**.** *Let X be the d-dimensional complete complex on n vertices. Then for all $0 \leq i < d$, $h^i > 1$.*

It is assumed here that the complete complex is accompanied with the uniform measures on each $X(i)$, so all weights and distances are with respect to the uniform distribution.

Proof. Firx $f \in C_i \setminus B^i$ such that $\mathrm{wt}(\delta f) = \varepsilon$. We will show that $\mathrm{dist}(f, B^i) < \varepsilon$. We will do so by finding g such that $\delta g \approx f$. By the assumption that $\mathrm{wt}(\delta f) = \varepsilon$ we have

$$\varepsilon = \Pr_{v_0,\ldots,v_{i+1}}[\delta f(v_0,\ldots,v_{i+1}) = 0] = E_{v_0} \Pr_{v_1,\ldots,v_{i+1}}[\delta f(v_0,\ldots,v_{i+1}) = 0].$$

Fix some $v_0 \in X(0)$ for which the above probability is at most the expectation ε. Define an $i-1$-chain g by $g(\tau) = f(v_0 \cup \tau)$ whenever $v_0 \notin \tau$, and otherwise $g(\tau) = 0$. This is very similar to our proof of Claim 1.3 above. We calculate the distance of f and δg :

$$\delta g(\sigma) = \sum_{x\in\sigma} g(\sigma \setminus \{x\}).$$

If $v_0 \notin \sigma$ this is equal to

$$= \sum_{x\in\sigma} f(\{v_0\} \cup \sigma \setminus \{x\})$$

which equals $f(\sigma)$ whenever $\delta f(\{v_0\} \cup \sigma) = 0$, which happens with probability exactly $1 - \varepsilon$.

Moreover for any $\sigma = \{v_0\} \cup \tau$ (where τ is an $i-1$-face that doesn't contain v_0), by definition

$$\delta g(\sigma) = g(\tau) + \sum_{x\in\tau} g(\{v_0\} \cup \tau \setminus \{x\}) = f(\{v_0\} \cup \tau) + 0 = f(\sigma).$$

So the probability that $\delta f(\sigma) \neq g(\sigma)$ is strictly smaller than ε, and so we have proven that $\mathrm{dist}(f, B^i) < \mathrm{wt}(\delta f)$. □

Question: where in this proof did we use the fact that X is the complete complex?

Notice that the argument is one of "defect correction". We are given f with small defect ($\mathrm{wt}(\delta f)$ is small), and we use it to construct a "corrected version" δg which is close to f and has no defect (since $\delta(\delta g) = 0$).

4 A Property Testing Perspective

Lubotzky and Kaufman observed that coboundary expansion can be viewd as a certain property testing result. In property testing, a property is a set $P \subset \{0,1\}^n$ and the goal is to find a so-called tester for this property. The tester is randomized, and reads a few bits from the string to-be-tested, and either accepts or rejects. The completeness is the probability of accepting a string $s \in P$. The soundness is the probability of accepting a string s that is far from P. One way to argue soundness is to show that if the tester rejects with probability less than ε then the string must be $c \cdot \varepsilon$ close to P.

Let us see how this relates to coboundary expansion. The property we are looking at is a property of i-chains, that of being a coboundary. Namely

$$P = B^i \subset \{0,1\}^{X(i)}.$$

There are potentially various testers for this property, but coboundary expansion talks about one specific test, the cocycle test. Fix an i-chain f to-be-tested. The tester is the following random procedure:

- Choose a random $i+1$-dimensional face σ
- Accept iff $\sum_{x\in\sigma} f(\sigma \setminus \{x\}) = 0 \mod 2$

In other words, the test checks that $\delta f(\sigma) = 0$ for a random σ. Clearly, the rejection probability of this test is exactly $\mathrm{wt}(\delta f)$. Coboundary expansion guarantees that if this probability is small then the distance of f from B^i is small,

$$\mathrm{dist}(f, B^i) \leq \frac{1}{h^i} \cdot \mathrm{wt}(\delta f).$$

Let us conclude with a few remarks.

- In property testing, the property is at the center of focus, and the exact test is less important. In (coboundary) expansion, the complex describes both the property and the test, so we are not interested in the general testability of B^i but in the question of whether the cocycle test is a good test.
- One can define an appropriate notion of expansion even in case of non-zero cohomology. Instead of coboundary expansion this would be cocycle expansion, and this is known as cosystolic expansion. It tests the property Z^i (which coincides with B^i only when the cohomology vanishes). Cosystolic expansion is defined to be

$$\tilde{h}^i = \min_{f \in C_i \setminus Z^i} \frac{\mathrm{wt}(\delta_i f)}{\mathrm{dist}(f, Z^i)}.$$

5 Locally Testable Codes

In the last part of the lecture we described the notion of locally testabe codes (LTCs). A locally testable code is an error correcting code $C \subset \{0,1\}^n$ that has a tester in the property-testing sense described above. The first example of an LTC is the Hadamard code $H \subset \{0,1\}^{2^k}$ whose codewords are the so-called "linear functions" $\ell_a : \{0,1\}^k \to \{0,1\}$. For each $a \in \{0,1\}^k$ the function ℓ_a is defined by $\ell_a(x_1, \ldots, x_k) = \sum_{i=1}^k a_i x_i \mod 2$.

$$H = \{\ell_a : \{0,1\}^k \to \{0,1\} \ : \ a \in \{0,1\}^k\}.$$

The testability of this code is also known as linearity testing: given a function $f : \{0,1\}^k \to \{0,1\}$, we want to test if $f \in H$. Namely, if there is some ℓ_a such that $f = \ell_a$.

The linearity test is simple, and makes only three queries to f:

- Choose $x, y \in \{0,1\}^k$ uniformly at random
- Accept iff $f(x) + f(y) = f(x+y)$

It is easy to see that if $f = \ell_a$ for some a then the test succeeds with probability 1. This establishes (so-called perfect) completeness. Soundness is proven through the following lemma

Lemma 5.1. *If* $\Pr[$ *Test fails* $] < \varepsilon$ *then there is some a such that* $\mathrm{dist}(f, \ell_a) < \varepsilon$.

Note how similar this statement is to the proof of Theorem 3.1 where we showed that if the cocycle test rejected f with probability at most ε then $\mathrm{dist}(f, \delta g) < O(\varepsilon)$ for some coboundary δg.