High Dimensional Expanders

Lecture 10: Boolean Analysis on High Dimensional Expanders

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Our goal in this talk is to find a "Fourier decomposition" for $\ell_2(X(k))$. This term is a bit vague, and not properly defined. If we only want an orthogonal decomposition - we can decompose $\ell_2(X(k))$, in numerous ways. If we want it to respect some operation of the group, then this restricts us to simplicial complexes that have some group operating on them. So, to better understand what kind of decomposition we can get, let's consider the more familiar Boolean hypercube.

1 The Boolean Hypercube

1.1 Basics

Recall that the Boolean hypercube is the graph $G = (V, E)$ where

$$V = \{0,1\}^n$$

$$E = \{\{x,y\} : x - y = e_i \pmod 2, \text{ for some } 1 \le i \le n\},$$

where e_i are the vectors of the standard basis. In other words $x \sim y$ if the differ by one coordinate.

We saw that the characters

$$\{\chi_S(x) = (-1)^{\sum_{i\in S} x_i} : S \subseteq [n]\}$$

are an orthonormal basis for the real valued functions on the vertices $\ell_2(V)$.

On the one hand, this basis has a lot of nice properties that make it useful: they are eigenvectors of the adjacency operator along with other natural operators (such as the *noise operator*, the *influence operator*), they behaive well under multiplication with one another.

Many of their properties only rely on the decomposition to "Fourier-levels". That is, if

$$f(x) = \sum_{S \subset [n]} \hat{f}(S)\chi_S(x).$$

then we can also write

$$f(x) = \sum_{j=0}^{n} f^{=j}(x)$$

where the jth-"Fourier level" is

$$f^{=j} = \sum_{S\subset[n],|S|=j} \hat{f}(S)\chi_S(x).$$

The adjacency operator of the Boolean hypercube acts separately on each level. More specifically:

$$A_G f^{=j} = (1 - \frac{2j}{n}) f^{=j}.$$

One the other hand, the characters have combinatorial meaning. One example for this is the FKN theorem, [FKN02]: a function whose weight is concentrated in the first level is close (in ℓ_2 or probabilisticly) to a dictatorship function.

Theorem 1.1 (Freidgut-Kalai-Naor)**.** *Let $\varepsilon > 0$, and let $f : \{0,1\}^n \to \{\pm 1\}$ be a Boolean valued function, so that*

$$\left\|f^{>1}\right\|^2 = \|f\|^2 - \left\|f^{=0} + f^{=1}\right\|^2 \leq \varepsilon.$$

Then there exists some dictator function $g \in \{\pm 1, \pm x_i\}$, so that

$$\|f - g\|^2 = O(\varepsilon).$$

Equivalently $Pr[f \neq g] = O(\varepsilon)$.

We see in this example that the "Fourier-levels" are what's important, and the specific coefficients to each character don't matter.

There are many characterizations of functions due to their Fourier levels. Our goal in this lecture is to try and carry this theory on to the high dimensional expander setting.

2 Boolean Analysis on Simplicial Complexes

High dimensional expanders are a much broader class of objects. We don't have any canonical group operating on them, or a natural symmetric structure. Still, we would like to construct a "Fourier decomposition" that has as much of the properties above as we can. But first, let's recall some of the things we saw previously in the course.

2.1 Quick Recap

Let X be a pure d-dimensional simplicial complex and $\pi_d : X(d) \to [0,1]$ a probability distribution. We define a stochastic process $\{S_j\}_{j=0}^d$ and measures $\pi_d, \pi_{d-1}, \ldots, \pi_0$ as follows:

- $S_d \sim pi_d$ is choosing some d-face.
- Given the choice of $S_{i+1} = \tau$, we choose $S_i \in X(i)$ by taking a vertex $v \in \tau$ uniformly at random and setting $S_i = \tau \setminus \{v\}$.

The measures $\pi_i : X(i) \to [0,1]$ are the marginals of S_i.

We Also defined the up and down operators

$$U_{\nearrow i+1} : \ell_2(X(i)) \to \ell_2(X(i+1));\ D_{\searrow i} : \ell_2(X(i+1)) \to \ell_2(X(i)),$$

for all $-1 \leq i \leq d-1$, by

$$U_{\nearrow i+1} f(s) = \mathop{\mathbb{E}}_{t \in X(i)\ t \subset s}[f(t)],$$

$$D_{\nearrow i} g(t) = \mathop{\mathbb{E}}_{s \in X(i+1)\ s \supset t}[g(s)],$$

and saw that $D^* = U$.

We abbreviate by

$$U_{j\nearrow i} = U_{\nearrow i} U_{\nearrow i-1} \ldots U_{\nearrow j+1},$$

similarly $D_{i\searrow j}$. It is convenient in formulas to denote the identity as $U_{i\nearrow i}, D_{i\searrow i}$ (of course, this is just notation).

The composition UD and DU are the adjacency operators of the random walks (which we can think of as graphs) on $X(j)$:

- The composition $D_{j+1\searrow j} U_{j\nearrow j+1}$ is the upper walk: given $\sigma \in X(j)$, we choose a face $\tau \in X(j+1)$ that contains it, and then choose a face $\sigma' \in X(j)$ that is contained in τ.

- The composition $U_{j-1\nearrow j}D_{j-1\searrow j}$ is the lower walk: given $\sigma \in X(j)$, we choose a face $\rho \in X(j-1)$ that is contained in σ, and then choose a face $\sigma' \in X(j)$ that is contains in ρ.

We previously saw that

$$D_{j+1\searrow j}U_{j\nearrow j+1} = \frac{1}{j+2}I + \frac{j+1}{j+2}M_j^+$$

where I is the identity, and M_j^+ is the non-lazy version of the upper random walk on the j-faces.

Our random-walk based definition to a γ-HDX was the following:

Definition 2.1 (High Dimensional Expander)**.** *Let X be a d-dimensional simplicial complex, $\gamma < 1$. We say that X is a γ-HDX if*

$$\left\| U_{j-1\nearrow j}D_{j-1\searrow j} - M_j^+ \right\| \leq \gamma$$

for all $0 \leq j \leq d-1$.

In the next couple of sections we'll see that the fact that the upper walk and lower walk are similar (up to laziness), will allow us to connect the different ℓ_2 spaces of the faces of $X(j)$, and decompose a function to its levels.

2.2 Decomposition to Fourier-levels

Spoiler 2.2. *Our decomposition for functions $f : X(k) \to \mathbb{R}$ is going to be*

$$f = \sum_{j=-1}^{k} f^{=j}$$

where $f^{=j} = U_{j\to k}h^{=j}$ for some specific $h^{=j}$. This is a decomposition to "approximate eigenspaces" of the upper (and lower walks).

For this discussion we fix some $k \leq d$. We would like to decompose $\ell_2(X(k))$ to its Fourier-levels. But when do we say a function is from level m?

In the Boolean hypercube case, this is easy. m-level functions are just linear combinations of the characters $\{\chi_S : |S| = m\}$.

One way to try and define an m-level function in $\ell_2(X(k))$ is a function

$$f = U_{m\nearrow k}h$$

for some function $h : X(m) \to \mathbb{R}$. However, this is not a partition. An m-level function by this definition may also be an $m-1$-level function (since h itself may be $h = Ug$). Still, this gives us intuition that a function is of levels $\leq m$ if it is in the image of $U_{m\nearrow k}$.

We fix the problem in the following sense.

Proposition 2.3 (Decomposition to Fourier levels)**.** *Let X be a d-dimensional simplicial complex, and let $k \leq d$. Let $f : X(k) \to \mathbb{R}$ be some function. Then there exists a decomposition*

$$f = \sum_{j=-1}^{k} f^{=j}$$

where:

1. *$f^{=-1} = U_{-1\nearrow k}h^{-1}$ is some constant function.*

2. *$D_{\searrow d-1}f^{=d} = 0$.*

3. *For all* $0 \leq j < k$ *there exists some* $h^{=j} \in X(j)$ *so that* $f^{=j} = U_{j\nearrow k} h^{=j}$*. Furthermore* $D_{\nearrow j-1} h^{=j} = 0$*.*

Notice that the sum of the j-lower levels is precisely the part that comes from the image of $U_{j\nearrow k}$, so we can really understand this as a function from a lower level.

Proof. Recall that for any linear transformation S, $Ker(S) = Im(S^*)^{\perp}$. In particular $Ker(D_{\searrow k-1}) = Im(U)^{\perp}$. Thus

$$\ell_2(X(k)) = Ker(D) + Im(U),$$

and for any $f : X(k) \to \mathbb{R}$, we can write $f = f^{=d} + Uh$, where $h \in X(\ell_2(X(k-1))$ and $f^{=d} \in Ker(D)$. Now we proceed by induction on h. □

Notice that this proposition is weak in the sense that we don't even know if the functions $h^{=j}$ are unique.

Claim 2.4. *The* $h^{=-1}, ..., h^{=k}$ *in decomposition above are unique for all functions, if and only if the operators* U *are injective.*

Proof. Exercise. □

When the operators U are unique, we say that the simplicial complex is *proper.*

Exercise 2.5.

1. *Any* $(d+1)$*-partite simplicial complex, is not proper.*
2. *A graph (*1*-dimensional simplicial complex) is not proper, if and only if it has a bipartite connected component.*
3. *Find an example for a* d*-dimensional simplicial complex that is not proper, and not* $(d+1)$*-partite.*

Claim 2.6. *Let* X *be a* d*-dimensional simplicial complex, and suppose that* X *is a* γ*-HDX for* $\gamma < \frac{1}{d+1}$*. Then* X *is proper.*

Proof. Let $f \in X(j)$ be some non-zero function for some $j < d$. Then

$$\langle Uf, Uf \rangle = \langle DUf, f \rangle = \frac{1}{j+2} \langle f, f \rangle + \frac{j+1}{j+2} \langle M^+ f, f \rangle$$

$$= \frac{1}{j+2} \langle f, f \rangle + \frac{j+1}{j+2} \left(\langle (M^+ - UD) f, f \rangle + \langle UDf, f \rangle \right) \geq$$

$$\frac{1}{j+2} \langle f, f \rangle + \frac{j+1}{j+2} \langle Df, Df \rangle - \frac{j+1}{j+2} \gamma \langle f, f \rangle .$$

Since $\langle Df, Df \rangle \geq 0$. and $\frac{1}{j+2} \geq \frac{1}{d+1}$ and $\gamma < \frac{1}{d+1}$ the required is positive. □

3 Orthogonality of Decomposition

In the next sections we'll show that the decomposition we gave above is "close" in some sense, to an orthogonal decomposition. We begin with the case of the complete complex.

3.1 The Complete Complex

Claim 3.1. *Let* X *be the* d*-dimensional completed complex on* n *vertices. Then*

$$\frac{i+2}{i+1} D_{i+1\searrow i} U_{i\nearrow i+1} - \frac{n-i}{n-i-1} U_{i-1\nearrow i} D_{i\searrow i-1} = \left(\frac{1}{i+1} - \frac{1}{n-i-1} \right) I$$

Proof. Note that In the complete complex case, the upper non-lazy operator and the lower non-lazy operator are the same:

$$(M_i^+)_{(\sigma',\sigma)} = (M_i^+)_{(\sigma',\sigma)} = Pr[\sigma'|\sigma] = \begin{cases} \frac{1}{n-i-1} & |\sigma \cap \sigma'| = i \\ 0 & otherwise \end{cases}.$$

Moreover, as the probability to stay in place in the complete complex also doesn't depend on the vertex, we also have that:

$$U_{i-1\nearrow i}D_{i\searrow i-1} = \frac{n-i-1}{n-i}M_i^+ + \frac{1}{n-i}I.$$

We recall that in any simplicial complex

$$D_{j+1\searrow j}U_{j\nearrow j+1} = \frac{1}{j+2}I + \frac{j+1}{j+2}M_j^+,$$

and the rest is direct calculation. □

Claim 3.2. *Let X be the complete complex. Let $f = U_{m\nearrow k}h$ be so that $D_{m\searrow m-1}h = 0$. Then*

$$DUf = \lambda_m^k f$$

$$UDf = \lambda_m^{k-1} f$$

where

$$\lambda_m^k = (1 - \frac{m+1}{k+2}) + o_n(1).$$

This claim immediately gives us orthogonality:

Corollary 3.3 (Orthogonality of Decomposition)**.** *Let X be the d-dimensional complete complex, and $f : X(k) \to \mathbb{R}$ be some function. Let $f = \sum_{j=-1}^{k} f^{=j}$ be its Fourier-level decomposition. Then for any $i \neq j$, $f^{=i} \perp f^{=j}$.*

This also gives us some insight regarding the non-expanding sets in the complete complex.

Corollary 3.4. *Let $f : X(k) \to \mathbb{R}$ an indicator of some set A. Suppose that $\left\|f^{\leq j}\right\|^2 < \varepsilon \left\|f\right\|^2$. Then*

$$\left\|DUf\right\|^2 \leq ((\frac{j+1}{k+2})^2 + \varepsilon) \left\|f\right\|^2.$$

The idea of the proof is inductive. The base case is will be direct, and then we can recursively calculate $DUU_{m\nearrow k}h$, by substituting DU with $\alpha UD + (1-\alpha)I$ (for appropriate α, given by Claim 3.1). Then we get

$$(1-\alpha)U_{m\nearrow k}h + \alpha UDU_{m\nearrow k}h,$$

and then again we substitue DU with $\alpha' UD + (1-\alpha')I$:

$$(1-\alpha)U_{m\nearrow k}h + \alpha U\left((1-\alpha')UDU_{m\nearrow k-1}h + \alpha' UDU_{m\nearrow k-1}h\right)$$

and so on. When we get to $U...UDh$, we get that $Dh = 0$. Thus we remain with $\lambda U_{m\nearrow k}h$ for an appropriate constant λ.

Proof. The proof goes by induction on k. For $k = 0$ note that if $f = Uh$ then f is constant, thus $UDf = DUf = 1f$.Otherwise, $f \in KerD$, and then $UDf = 0$, and $DUf = (\frac{1}{2} - \frac{1}{2(n-1)})f$ since this is the lazy version of the complete graph's adjacency operator.

Now assume this is true for $k-1$, and consider $f = U_{m\nearrow k}h$.

- Consider first the operator UD: If $f \in KerD$ the calculation is clear. Otherwise, to calculate $UDU_{m\nearrow k}h$, we use associativity and do $U(DU)U_{m-1\nearrow k-1}$. From the induction hypothesis, we get that this is $U(\lambda_{m-1}^{k-1}U_{m-1\nearrow k-1}$ and the equality follows.

- Now for DU: By Claim 3.1:

$$UD(U_{m\nearrow k}h) = U\left[\left(\frac{1}{k+2} - \frac{k+1}{(k+2)(n-k-1)}\right)I + \frac{(n-k)(k+1)}{(n-k-1)(k+2)}UD\right]U_{m\nearrow k}h =$$

$$\frac{1}{k+2}U_{m\nearrow k}h + \frac{k+1}{k+2}UD(U_{m\nearrow k}h) + \frac{1}{n-k-1}\left(-\frac{k+1}{k+2}I + \frac{k+1}{k+2}UD(U_{m\nearrow k}h\right)$$

From what we saw above, $UD(U_{m\nearrow k}h) = \lambda_{m-1}^{k-1}U_{m\nearrow k}h$ is an eigenvector, and so we get:

$$= \left(\frac{1}{k+2} + \frac{k+1}{k+2}(1 - \frac{m-1}{k+1}) + o_n(1)\right)U_{m\nearrow k}h = \lambda_m^k U_{m\nearrow k}h.$$

$\square$

3.2 High Dimensional Expanders

Reflecting on the previous proof, what we actually needed, is the fact we could write

$$DU = \alpha UD + \beta I.$$

In high dimensional expanders, we only have an approximate equality, that is:

$$\left\| DU - \frac{j+1}{j+2}UD - \frac{1}{j+2}I \right\| \leq \gamma.$$

This is true because

$$DU = \frac{j+1}{j+2}M^+ + \frac{1}{j+2}I$$

and

$$\left\| M^+ - UD \right\| \leq \gamma.$$

In the approximate case, the upper and lower operators do not commute in general, thus we can't expect both operators to have the same eigenspaces. However, we can say that the $f^{=j}$'s are *approximate* eigenvectors. That is,

$$\left\| DUf^{=j} - \lambda_j^k f^{=j} \right\| = O_d(\gamma)\left\| f^{=j} \right\|.$$

In particular, we'll get that this decomposition is approximately orthogonal. When $j_1 \neq j_2$,

$$\left\langle f^{=j_1}, f^{=j_2} \right\rangle = O_d(\gamma)\left\| f^{=j_1} \right\| \left\| f^{=j_2} \right\|.$$

Theorem 3.5 (Approximate Eigenvector Decomposition)**.** *Let $f = U_{m\nearrow k}h$ for some $h \in kerD_{k\searrow k-1}$. Then*

$$\left\| DUf - \lambda_m^k f \right\| \leq (k-m)O(\gamma).$$

$$\left\| UDf - \lambda_{m-1}^{k-1} f \right\| \leq (k-m)O_d(\gamma)\left\| f \right\|.$$

We abuse the notation and write $g_1 = g_2 + O(\gamma g_3)$.

For example, in that graph case, when we look at functions on the vertices, we can separate them to their constant part where $UDf^{=-1} = 1f^{=-1}$, and to the part that is orthogonal to the constant part where

$$\left\| UDf^{=0} - \frac{1}{2}f^{=0} \right\| = O(\gamma) \left\| f^{=0} \right\|.$$

Again,

Proof. We begin proving something easier:

Claim 3.6.

$$\left\| DUf - \lambda_m^k f \right\| \leq (k-m)O(\gamma) \left\| h \right\|.$$

$$\left\| UDf - \lambda_{m-1}^{k-1} f \right\| \leq (k-m)O_d(\gamma) \left\| h \right\|.$$

Notice that in this claim, we compare ourselves to the norm of h. Since U is a contracting operator, we are allowing ourselves more error.

Afterwards we'll show that the norm of h and the norm of f are proportionate up to a constant factor that depends only on the dimension of the simplicial complex:

Claim 3.7 (Equivalent Norms)**.** *There exists a global constant ρ_k, that depends only on k (but not on any other parameters of the simplicial complex), s.t.*

$$\left\| U_{k \nearrow m} h \right\| \geq \rho_d (1 \pm O(\gamma)) \left\| h \right\|.$$

Proof of Claim 3.6. This proof for Claim 3.6 follows the same steps of the proof for the complete complex. We use induction on k where the case for $k = 0$ is clear.

Assume for $k-1$, and consider $f = U_{m \nearrow k} h$.

$$UDf = U[(DU)U_{m \nearrow k-1} h] = U[\lambda_{m-1}^{k-1} U_{m-1 \nearrow k-1} h + (m-k-1)O(\gamma h)] = \lambda_{m-1}^{k-1} U_{m \nearrow k} h + (m-k-1)O(\gamma h).$$

The last equality is due to the fact that U is contracting (that is $\|Ug\| \leq \|g\|$).

Calculating for DU:

$$DUU_{m \nearrow k} h = \frac{1}{k+2} U_{m \nearrow k} h + \frac{k+1}{k+2} UDU_{m \nearrow k} h + O(\gamma U_{m \nearrow k} h)$$

As before, we use the induction hypothesis on $k-1$ and get:

$$= \frac{1}{k+2} U_{m \nearrow k} + \frac{k+1}{k+2} U(\lambda_m^{k-1} U_{m \nearrow k-1} h + (m-k-1)O(\gamma h)) + O(\gamma h)$$

$$= \lambda_m^k U_{m \nearrow k} h + (m-k)O(\gamma h).$$

□

Proof Sketch of Claim 3.7. the proof here is also by induction.

$$\langle f, f \rangle = \langle U_{m \nearrow k} h, U_{m \nearrow k} h \rangle = \langle DUU_{m \nearrow k-1} h, U_{m \nearrow k-1} h \rangle = \lambda_m^{k-1} \langle U_{m \nearrow k-1} h, U_{m \nearrow k-1} h \rangle + \langle O(k\gamma h), h \rangle$$

By the induction hypothesis we can write

$$= \lambda_m^{k-1} \rho_{k-1} \langle h, h \rangle + \langle O(k\gamma h), h \rangle$$

and by Cauchy-Schwartz we get (in both directions)

$$= (\lambda_m^{k-1} \rho_{k-1}) \langle h, h \rangle + O_k(\gamma) \langle h, h \rangle$$

and the claim follows. □

Combining the two claims, and the theorem follows. □

Remark 3.8. 1. *We won't get in to it today, but this decomposition also has combinatorial meaning. For example, there is an FKN theorem for high dimensional expanders, that shows that the Fourier levels measure closeness to a degree one function.*

2. *Another thing we won't see today, is that one can use this to extend the expander mixing lemma to higher dimension.*

3. *Finally, notice that what we really needed is the stochastic process structure for this proof, i.e. the random variable $S = (S_d, ..., S_0)$. One can actually extend this theory to other structures with such a process. An example for this is the Grassmann POSet $X = Gr_q(n, d)$ where*

$$\forall i = -1, 0, 1, ..., d \; X(i) = \{W \subset F_q^n : dim W = i + 1\}$$

and our process is to choose S_d uniformly at random, and then choose a flag *of subspaces inside it.*

References

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