High Dimensional Expanders

Lecture 13: Double Samplers and HDX

Instructor: Irit Dinur *Scribe: Omer Lavy*

1 Samplers

In this chapter we discuss double samplers. We will introduce the notion of double samplers and then show how one can use HDX in order to construct efficient double samplers. In the first section we will present samplers which are easier to construct and are related to expander graphs (i.e. one dimensional HDX).

Recall that a sampler is a random algorithm. The sampler is defined by two parameters:

1. The sampler's accuracy ϵ.
2. The confidence (error) parameter δ.

The input of the sampler is a function $f : [n] \to [0, 1]$ where $[n]$ denotes, as usual the set $\{1, ..., n\}$. The sampler probes d points $x_1, ..., x_d$ such that:

$$Pr\left[\left|\frac{1}{d}\sum_i f(x_i) - \mathop{\mathbb{E}}_{x\in[n]} f(x)\right| > \epsilon\right] < \delta$$

where $\mathop{\mathbb{E}}_{x\in[n]} f(x) = \frac{1}{n}\sum_{i=1}^n f(x_i)$ denotes the the expectation of f. A good sampler runs in short time related to n. We now describe a model for a sampler. The random process is given by r coins flip. This results in $m = 2^r$ different states which we denote by m vertices. The set of those is denoted as V_1. In each state d out of n points are being sampled. We have then another set of vertices, which we denote by V_0. We connect to each state vertex d vertices from V_0 which corresponds to the points sampled in this state. This give rise to the following definition of a sampler graph which will be used in the sequel.

Definition 1.1. *A d -regular bipartite graph $G(V_0, V_1, E)$ is called a (ϵ, δ) sampler if for every f, $f : [n] \to [0, 1]$, the following inequality holds:*

$$\mathop{Pr}_{u\in V_1}\left[\left|\mathop{\mathbb{E}}_{x\sim u} f(x) - \mathop{\mathbb{E}}_{x\in V_0} f(x)\right| > \epsilon\right] < \delta$$

Samplers are closely related to expanders graphs:

Proposition 1.2. *Let $G(V_0, V_1, E)$ be bipartite graph, and suppose that G is a λ expander, then G is a $(\epsilon, \frac{\lambda^2}{\epsilon})$ sampler.*

Remark 1.3. *There is a converse to the previous proposition which is also true: If G is a (ϵ, δ) sampler then G contains a subgraph which is an expander. for details see Theorem 2.18 in [DHK+18].*

Before we give a proof we recall the defnition of a bipartite expander. Recall that given a bipartite graph $G(V_0, V_1, E)$, we define an operator $M : l^2(V_0) \to l^2(V_1)$ as

$$Mf(u) = \mathop{\mathbb{E}}_{[x,u]\in E} f(x).$$

When restricted to the 1- dimensional space of constant functions this is an operator of norm 1. Recall further that the orthogonal complement of this is space is the space of functions with average 0. The norm of the operator when restricted to the orthogonal complement of the space of constant functions is the expansion of the graph. Namely

$$\lambda = \sup\{||Mf||_{V_1} \; s.t. \; ||f||_{V_0} = 1 \; and \; \mathop{\mathbb{E}}_{x\in V_0} f(x) = 0\}$$

Proof of Proposition 1.2: Fix $f \in l^2(V_0)$, and let $f' = f - \mathop{\mathbb{E}}_{V_0} f$ be its orthogonal projections on the complement of the subspace of constant functions. Let $T = \{u \in V_1 \ s.t.\ ||Mf(u) - \mathop{\mathbb{E}}_{V_0} f|| > \epsilon\}$. Then T is just the set of points in V_1 that fails in predicting f. Then as G is λ expander, $||Mf(u) - \mathop{\mathbb{E}}_{V_0} f||^2 = ||Mf'||^2 < \lambda^2 ||f||^2 \leq \lambda^2$. Where the last inequality holds since $||f|| \leq \max f \leq 1$. Then by Markov inequality:

$$Pr(T) = Pr\left[Mf'(u))^2 > \epsilon^2\right] < \frac{\lambda^2}{\epsilon^2}$$

□

2 double samplers

Double sampler is a generalization of a sampler. For details about double samplers and their applications see [DHK+18]. Recall that a 3- layer graph $G(V_0, V_1, V_2, E)$ is a graph in which the restrictions $G(V_0, V_1)$, $G(V_1, V_2)$ are bipartite graphs and with no edges between V_0 and V_1.

Definition 2.1. *A 3- layer graph $G(V_0, V_1, V_2, E)$ is a double sampler if it satisfies the following conditions:*

1. *$G(V_0, V_1)$ is a λ expander.*
2. *$G(V_1, V_2)$ is a λ expander.*
3. *For a vertex $T \in V_2$ denote V_1^T the neighbors of T in V_1, and by V_0^T their neighbors in V_0. $G(V_0^T, V_1^T)$ is a λ expander, for every vertex $T \in V_2$*

Example 2.2. *We give now examples of double samplers. The fact that those are really double expanders will follow from Theorem 3.1.*

1. *A dense double sampler. For some $0 << k << d << n$, let*
   - *$V_0 = \{1, \ldots, n\} = [n]$.*
   - *$V_1 = \binom{[n]}{k}$, i.e. the vertices in V_1 represents subsets of $[n]$ of size k.*
   - *$V_2 = \binom{[n]}{d}$, i.e. the vertices in V_2 represents subsets of $[n]$ of size d.*

   Then:
   - $\lambda^2(G(V_0, V_1)) = \frac{1}{k+1} + o(1)$.
   - $\lambda^2(G(V_1, V_2)) = \frac{k+1}{d+1} + o(1)$.
   - $\lambda^2(G(V_0^T, V_1^T)) \leq max\left(\frac{1}{k+2}, \frac{1}{d-k+2}\right) + o(1)$*, for every $T \in V_2$.*

   So G is a double sampler with $\lambda^2 \leq max\left(\frac{1}{k+2}, \frac{k+1}{d+2}, \frac{1}{d-k+2}\right) + o(1)$.

2. *The Grassmanian double sampler. Fix finite field $\mathbb{F}_q$, and let $V = \mathbb{F}_q^n$.*
   - *V_0 be the set of all 1-dim subspaces in V (i.e lines that pass through the origin in V).*
   - *V_1 be the set of all 2-dim subspaces in V.*
   - *V_2 be the set of all 3-dim subspaces in V.*

   As we did in the flag complex we connect vertices (from V_0 to V_1, and from V_1 to V_2) whenever there is containment. Then G is a double sampler with $\lambda \leq \frac{1}{\sqrt{q+1}}$.

Note that in the examples above the sets V_1 and V_2 grow rapidly with the size of V_0. For example in the flag complex, if we denote $|V_0| = N$, then $|V_1| \sim N^2|$, and $|V_2| \sim N^3$. In the next section we will use HDX to construct an efficient double sampler.

3 HDX are double samplers

So far, the examples we gave of double samplers were of (not very sparse) HDX. We Recall the definition of a spectral HDX. For a simplicial complex X we define the U and D operators as follows. The U_i, $U_i : l^2(X(i)) \to l^2(X(i+1))$ operator is an operator averaging over the faces contained in a given face. Namely:

$$Uf(s) = \mathop{\mathbb{E}}_{t \sim s} f(t).$$

Similarly the D operator, $D_i : l^2(X(i)) \to l^2(X(i-1))$ is defined by:

$$Df(s) = \mathop{\mathbb{E}}_{t \sim s} f(t).$$

Note further that $U_i^* = D_{i+1}$. We defined the (lazy) random walk to be $L_i^+ = UD$ and denoted the non-lazy random walk as M_i^+. The simplicial complex X is a λ expander if for all $i < d$,

$$\left\| M_i^+ - L_i^+ \right\| \leq \lambda.$$

We now show how HDX can be used in order to construct double samplers. Let X be a γ-HDX, and define:

- $V_0 = X(0)$.
- $V_1 = X(k)$.
- $V_2 = X(d)$.

As usual we connect vertices whenever there is containment. The main result of this section is that this defines a double sampler, namely:

Theorem 3.1. *The graph defined above is a λ double sampler with $\lambda \leq max\left(\frac{k+1}{d+1}, \frac{1}{k+1}, \frac{1}{d-k+2}\right) + o(\gamma)$*

Remark 3.2. *Example 1 is then a special case of this Theorem. This sampler corresponds to the complete complex which is a HDX. This HDX however is not sparse. The theorem above ensures existence of double samplers with $o(n)$ vertices and $\lambda \to 0$.*

Our goal now is to prove Theorem 3.1. our first step is the following easy claim which will be used later:

Claim 3.3. *Let $G(V_0, V_1, V_2)$ be a 3-layer graph, on $V_0 \sqcup V_1 \sqcup V_2$. Denote:*

- $\lambda_1 = \lambda\left(G(V_0, V_1)\right)$.
- $\lambda_2 = \lambda\left(G(V_1, V_2)\right)$.

then $\lambda\left(G(V_0, V_2)\right) \leq \lambda_1 \lambda_2$

Proof. Fix $f \in l^2(V_0)$ with $\mathop{\mathbb{E}}_{V_0} f = 0$. Note that $\mathop{\mathbb{E}}_{V_1} Uf = 0$. Indeed,

$$\mathop{\mathbb{E}}_{V_1} Uf = \langle Uf, 1\rangle_{V_1} = \langle f, U^* 1\rangle_{V_0} = \langle f, 1\rangle_{V_0} = \mathop{\mathbb{E}}_{V_0} f = 0.$$

Hence $\mathop{\mathbb{E}}_{V_2} U_2 U_1 f \leq \lambda_2 U_1 f \leq \lambda_2 \lambda_1 f$. □

The following lemma which we prove later, will play a major roll in the proof of Theorem 3.1.

Lemma 3.4. *For the graph described above,*

$$\lambda^2\left(G(X(i), X(i+1))\right) \leq \frac{i+1}{i+2} + i(\gamma). \tag{3.1}$$

Corollary 3.5. *For the graph described above, and any $0 \le i < j \le d$,*

$$\lambda^2\left(G(X(i), X(j))\right) \le \frac{i+1}{j+1} + \underset{i,j}{o}(\gamma).$$

In particular,

1. $\lambda^2\left(G(X(k), X(d))\right) \le \frac{k+1}{d+1} + \underset{k,d}{o}(\gamma)$.
2. $\lambda^2\left(G(X(0), X(k))\right) \le \frac{1}{k+1} + \underset{0,k}{o}(\gamma)$.

We give a proof for the case $\gamma = 0$. For the general case, one should open the brackets carefully and omit square factors on the way (we have to assume here $\gamma << 1$).

Proof. Indeed, using claim 3.3 inductively we get,

$$\lambda^2\left(G(X(i), X(j))\right) \le \lambda^2(U_i)\lambda^2(U_{i+1}) \dots \lambda^2(U_{j-1}).$$

Using Lemma 3.4 we have then:

$$\lambda^2\left(G(X(i), X(j))\right) \le \frac{i+1}{i+2} \cdot \frac{i+2}{i+3} \cdot \dots \cdot \frac{j}{j+1} = \frac{i+1}{j+1}$$

□

proof of Lemma 3.4. The proof is by induction. Assume first that $i = 0$. Note that since $D_1 = U_0^*$, we have that $\lambda^2(U_0) = \lambda(D_1U_0)$. The upper random walk, D_1U_0 is a $\frac{1}{2}$ lazy so $D_1U_0 = \frac{1}{2}\mathbb{I}d + \frac{1}{2}L_0^+$. As X is γ expander, $\left\|L_0^+ - U_{-1}D_0\right\| \le \gamma$. Let $f \in l^2(X(0))$ then be function orthogonal to the constants. Then $U_{-1}D_0f = 0$, hence,

$$\|L_0 f\| = \|L_0 f - U_{-1}D_0 f\| \le \gamma \|f\|.$$

finally,

$$\|D_1U_0f\| = \left\|\left(\frac{1}{2}\mathbb{I}d + \frac{1}{2}L_0^+\right)f\right\| \le \frac{1}{2}\left(\|f\| + \left\|L_0^+ f\right\|\right) \le \frac{1}{2}\|f\|(1+\gamma).$$

hence:

$$\lambda(D_1U_0) \le \frac{1}{2}(1+\gamma).$$

Now assume $i > 0$. Now

$$D_{i+1}U_i = \frac{1}{i+2}\mathbb{I}d + \frac{i+1}{i+2}L_i^+.$$

As X is γ expander, $\left\|L_0^+ - U_{i-1}D_i\right\| \le \gamma$. hence, for $f \in l^2(X(i))$,a function which is orthogonal to the constants,

$$\left\|(L_i^+ - U_{i-1}D_i)f\right\| \le \gamma \|f\|$$

by triangle inequality:

$$\left\|L_i^+ f\right\| - \|U_{i-1}D_i f\| \le \gamma \|f\|$$

and

$$\left\|L_i^+ f\right\| \le \|U_{i-1}D_i f\| + \gamma \|f\|.$$

Assuming that 3.1 holds for every $j < i$ we have then,

$$\left\|L_i^+ f\right\| \le \|f\| \left(\frac{i}{i+1} + \gamma i + \gamma\right).$$

Then

$$\|D_{i+1}U_i\| \leq \frac{1}{i+2} + \frac{i+1}{i+2}\left(\frac{i}{i+1} + \gamma i + \gamma\right) = \frac{i+1}{i+2} + \gamma(i+1).$$

□

Next we study the family of graphs, $G(V_0^T, V_1^T)$. The following observation is immediate.

Claim 3.6. *For all $T \in V_2$, G^T is the graph vertices Vs. k- sets of the complete complex on $d+1$ vertices.*

Corollary 3.7. *For all $T \in V_2$, $\lambda^2(V_0^T, V_1^T) \leq \frac{1}{d+1-k}$.*

Proof. still needs a proof □

Theorem 3.1 now follows easily.

Proof of Theorem 3.1. Indeed by Corollary 3.5, we have:

1. $\lambda^2\left(G(X(k), X(d))\right) \leq \frac{k+1}{d+1} + \underset{k,d}{o}(\gamma)$.
2. $\lambda^2\left(G(X(0), X(k))\right) \leq \frac{1}{k+1} + \underset{0,k}{o}(\gamma)$.

By the Corollary above,

$$\lambda^2(G(V_0^T, V_1^T)) \leq max\left(\frac{1}{k+2}, \frac{1}{d-k+2}\right) + o(1), \text{ for every } T \in V_2.$$

Hence G is a double sampler. □

References

[DHK+18] Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. List decoding with double samplers. *CoRR*, abs/1808.00425, 2018.