Recall our definition of a spectral expander: \( |\lambda_2| \leq \rho \).

An "ultimate" expander has \( \lambda_1 = 1 \) and \( \lambda_2 = \ldots = \lambda_n = 0 \).

What does such a graph look like? If \( \Pi = \text{uniform}, \) this is the complete graph with self-loops.

For other \( \Pi, \) the corr. matrix is \( J_{\Pi} = \left( \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right) \)

\begin{itemize}
  \item check: \( \cdot J_{\Pi}^2 = 1 \)
  \item if \( f \perp \Pi \) (i.e., \( E_{\Pi} f = 0 \)) then \( J_{\Pi} f = 0 \).
\end{itemize}

This is an "ultimate" expander, but dense. It is surprisingly that it has a sparse approximation.

**Note:** \( J^2 = J \) (and this characterizes \( J \) if connected)

**Def:** Operator norm: \( M : l^2(V_1) \rightarrow l^2(V_2) \)
\[ ||M|| := \sup_{f \neq 0} \frac{||Mf||}{||f||} \]

**Claim:** \( ||M|| = \lambda_{\text{max}} \)

**Proof:** \( \forall f \neq 0 \quad f = \text{span} f \)
\[ \langle mf, mf \rangle = \lambda \langle f, f \rangle \]
\[ \lambda_{\text{max}} = \text{max} \lambda_i \]

Clearly, \( \langle mf, mf \rangle \) is maximized when \( f = f_{\text{max}} \).

**Claim:** \( ||A - J|| \leq \lambda_2 \)

**Proof:** it is not always clear how \( ||A - J|| \) relates to \( ||A||, ||J|| \), because they may not share an eigenbasis. However, \( A \) and \( J \) do.

\begin{align*}
A1 &= 1 & J1 &= 1 & (A - J)1 &= 0 \\
Af &= \lambda f & Jf &= 0 & (A - J)f &= \lambda
\end{align*}

Therefore \( \lambda_{\text{max}} (A - J) = \lambda_2 \).
Today we will study a new definition of HDX that is equiv to last time.

A RW on a graph can be described by:

1) start at a vertex $v_0$
2) at step $i=0$:
   - up) choose a random edge $e \in V$.
   - down) choose a random vertex $v \in V$, set $v_{i+1} = v$.

Notice that $\text{Prob}(v_{i+1} = v_i) = \frac{1}{2}$. This is called a **Lazy random walk**. A different variant of a random walk chooses $v \in V$ in the downstep to be distinct from $v_0$.

This is the non-lazy random walk.

To write the transition matrix, we need to calc the probability $\Pr(v_{i+1} | v_i)$

$$
\Pr(v_{i+1} | v_i) = \sum_{e \in \text{adj}(v_i)} \Pr(v_{i+1} | e) \Pr(v_i | e) = \frac{\Pr(v_{i+1} | v_i)}{\Pr(v_i)} \frac{1}{2} \quad \text{if } e \in \text{adj}(v_i)
$$

$$
\Pr(v_{i+1} | v_i) = \sum_{e \in \text{adj}(v_i)} \Pr(v_{i+1} | e) \Pr(v_i | e) = \frac{\Pr(v_{i+1} | v_i)}{\Pr(v_i)} \frac{1}{2} = \frac{1}{2}
$$

$$
T_{\text{upperRW}} = \begin{bmatrix}
\frac{1}{2} & X(v_i) & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
X(v_i) & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
$$

In terms of the transition matrix,

$$
T_{\text{lazierRW}} = \frac{1}{2} T_{\text{upperRW}} + \frac{1}{2} I_d
$$

$T_{\text{upperRW}}$ is the normalized adjacency matrix.
**RW in higher dimensions**

It is natural to look at a random process that walks from edge to edge.

1. **Start at an edge** $e_0$ at step $i = 0$.
   - **Up** choose a random triangle $t 
i e_i$.
   - **Down** choose a random edge $e 
i t$ and set $e_{i+1} = e$.

We can see in the example that the edge $RW$ is disconnected $(t_1, t_2, t_3)$. How lazy is this $RW$? $e_{i+1} = e_i$ w. prob $\frac{1}{3}$.

A different $RW$ can be obtained by first going down and then going up.

2. **Start at an edge** $e_0$ at step $i = 0$.
   - **Down** choose a random vertex $v 
i e_i$.
   - **Up** choose a random edge $e 
i v$ and set $e_{i+1} = e$.

We will see that an expander is when the non-lazy upper walk has lower walk.

**The Up and Down Operators**

The $RW_{up}$ can be decomposed into two steps: $x(i) \to x(i+1)$ and then $x(i+1) \to x(i)$. We explicitly describe these two steps.

This can be described in 3 ways.

Let $D : l_2(x(i)) \to l_2(x(i+1))$,

\[
Df(n) = \sum_{e \in \mathcal{E}} \text{Prob}(e|\mathcal{U}) \cdot f(e) = \mathbf{E} f(e)
\]

Let $\mathcal{U} : l_2(x(i+1)) \to l_2(x(i))$.
\[ U_g(e) = \frac{1}{2} \left( g(u) + g(v) \right) = \sum_{u \in e} g(u) \quad \text{assuming } e = \langle u, v \rangle \]

The transition matrix of the lower, upper-\( (\text{non-lazy, lazy}) \) walks can be expressed in form of \( U, D \):

**lower RW:** \( UD \)

\[ UD = \begin{bmatrix} x(0) \\ \hline \hline \hline \hline \hline \hline \end{bmatrix} = \begin{bmatrix} x(0) \\ \hline \hline \hline \hline \hline \hline \end{bmatrix} \]

indeed:

\[ UD \cdot \langle e, e \rangle = \sum_{v} U_{\langle e, v \rangle} D_{\langle v, e \rangle} = \sum_{v} P_{\langle v, e \rangle} B_{\langle e, v \rangle} \]

**upper RW:** \( DU \)

(lazy)

check similarly that 

\[ DU_{\langle e, v \rangle} = \sum_{v} D_{\langle v, e \rangle} U_{\langle e, v \rangle} \]

2. \( DU = \text{Id} \) (non-lazy)

**Observe**

suppose \( f \in \ell_{2}(x(0)) \) and \( g \in \ell_{2}(x(0)) \), then

\[ \langle f, Ug \rangle = \langle Du, g \rangle \]

indeed:

\[ \langle f, Ug \rangle = \sum_{e} \sum_{v} \langle f, e \rangle \sum_{v} \sum_{e} \langle U, g \rangle = \sum_{v} \sum_{e} \langle f, e \rangle \sum_{v} \sum_{e} \langle g, \rangle \]

\[ \langle Du, g \rangle = \sum_{v} \sum_{e} \langle Du, e \rangle \cdot g(v) = \sum_{v} \sum_{e} \langle f(e), \rangle \cdot g(v) \]

**Corollary:**

\[ \langle UDf, f' \rangle = \langle Du, f' \rangle = \langle f, UDf' \rangle \]

\[ \langle DUG, g' \rangle = \langle UG, g' \rangle = \langle g, DUG' \rangle \]

i.e. both \( UD \) and \( DU \) are self-adjoint.
Generalizing to higher dimension

\[ D_i : l_2(x(i+1)) \to l_2(x(i)) \]

\[ U_{n} = D_{n}^* : l_2(x(i)) \to l_2(x(i+1)) \]

The graph of \( D \times U \)

\[ \text{The matrix } D \]
\[ D[1], R(x(t)) \] set

Non lazy RW Let \( M^+ \) be the transition matrix of the non lazy RW.

We can calculate its entries through the identities

\[ DU_n = \frac{1}{2} \text{Id} + \frac{1}{2} M^+ \]

in higher dim, \( i \),

\[ DU_n = \frac{1}{i+2} \text{Id} + \frac{i+1}{i+2} M^+ \]

Define: A complex \( X \) is a \( \gamma \)-RW-expander if

\[ \forall \omega \in \partial \Omega \quad \| U_{D_i} - M_i^+ \| \leq \gamma \]

Example 1: the 1-dimensional case.

\[ f : X(0) \to \mathbb{R} \]
\[ U_{i+1} : X(i) \to \mathbb{R} \]
\[ D_{2-1} : X(-i) \to \mathbb{R} \]

(recall \( X(-i) = \{ x_i \} \))
What is $U_{D_{XY}} f (v)$? $< I, U_{D} f >$

$= < D_{1}, D_{f} > = E_{f}(w)$

$D_{XY} = \left\{ \begin{array}{cc}
T(v) & \text{if } T(v) \neq T(w) \\
\{v\} & \text{if } T(v) = T(w) \end{array} \right.$

$U_{X_{0}} = \left\{ \begin{array}{cc}
\pi & \text{if } \pi(v) = 1 \\
\{v\} & \text{otherwise} \end{array} \right.$

$U_{X_{c}} = \left\{ \begin{array}{cc}
\pi & \text{if } \pi(v) = 1 \\
\{v\} & \text{otherwise} \end{array} \right.$

Theorem: If $X$ is a $\gamma$-skeleton expander then it is a $\gamma'$-RW-expander.

Proof for dim $X = 2$.

- $\| U_{D_{XY}} - M_{+} \| \leq \gamma'$
- $\| U_{M_{XY}} D_{XY} - M_{+} \| \leq \gamma'$

The first item we already checked.

The second item:

Recall that $M_{XY}$ is a $\gamma'$-RW-expander if $< A(f), f > \leq \gamma' \| f \|$ for all $f$.

Claim: $< A_{f} f > \leq \gamma' \| f \|$. Clearly, $< A(f), f > \leq M_{XY} \| f \| \leq \gamma' \| f \|$. For the converse, write $f = \sum \xi_{i} \eta_{i}$ where $\eta_{i}$ is the eigenvector of $M_{XY}$ corresponding to $\lambda_{i}$.

$< A_{f}, f > = < A(\xi_{i}, \eta_{i}), \sum \xi_{i} \eta_{i} >$

$= \sum \xi_{i} \eta_{i} \leq \gamma' \sum \xi_{i} \eta_{j} \leq \gamma' \| f \|$.
\[ \langle \text{udf}, f \rangle = \langle \text{df}, df \rangle = \langle \text{df}(e) - df'(e) \rangle \]

\[ \langle M^+ f, f \rangle = \sum_{e_1, e_2, v, u} E \langle f(u) f(v) \rangle \]

Finally, we use the link's expansion!

\[ \langle M^+ - \text{udf}, f \rangle \leq \gamma \langle f, f \rangle \]

so \[ \| M^+ - \text{udf} \| \leq \gamma \].

In this proof again we expressed the global spectrum through the local links.

\[ a \leq \sum_{x} \lambda_i \]

and since \[ (x, f)^2 = \langle x, x \rangle \] this is a weighted avg of the \( \lambda_i \)'s, so \[ \lambda_i = \lambda_{max} \].

Converse theorem: If \( X \) is a \( d \)-RW expander then it is a \( f \) link expander.

**Proof:** Fix \( d=2 \). Fix \( v \in X(o) \) and let us study \( \lambda(X_v(o), X_v(i)) \).

Suppose \( h : X_v(i) \longrightarrow \mathbb{R} \) is a non-constant eigenfunction.

(so \( \sum_{u \sim X_v(o)} h(u) = 0 \)).

Define \( f : X(o) \longrightarrow \mathbb{R} \) by \( f(x) = \sum_{a = v} h(a) \).

\[ E f(x) = 2 \prod_{u} v \neq x(u) = 0 \]

\[ E f(x) = 2 \prod_{u} v \neq x(u) = 0 \]
Assume wlog that \( \|f\| = 1 \). (now multiply by \( 1/f \))

\[
\|h\|^2 = \langle h, h \rangle_{X(\sigma)} = \mathbb{E} f(h_x)^2
\]

\[
\|f\|^2 = \langle f, f \rangle_{X(\sigma)} = \mathbb{E} f(e)^2 = 2\Pr(e) \cdot \mathbb{E} f(e)^2
\]

So

\[
\|f\|^2 = 2\Pr(e) \cdot \|h\|^2
\]

\[
\langle \Delta f, f \rangle = \langle \Delta f, \Delta f \rangle = \mathbb{E}_u \mathbb{E}_\varepsilon \mathbb{E}_e f(e) f(e_0) = \mathbb{E}_\varepsilon \mathbb{E}_e f(e) f(e_0) + \mathbb{E}_\varepsilon \mathbb{E}_e f(e) f(e_0)
\]

\[
\langle \Delta f, f \rangle = \mathbb{E}_\varepsilon \mathbb{E}_e f(e) f(e_0)
\]

\[
\langle M^2 f, f \rangle = \mathbb{E}_u \mathbb{E}_{h, h_0} f(h_0) f(h_u)
\]

\[
\langle M^2 f, f \rangle = \mathbb{E}_u \mathbb{E}_{h, h_0} f(h_0) f(h_u)
\]

\[
\gamma = \langle M^2 f, f \rangle - \langle \Delta f, f \rangle = \mathbb{E}_\varepsilon \mathbb{E}_e f(e) f(e_0) = \Pr(e) \cdot \mathbb{E}_\varepsilon f(e) f(e_0)
\]

\[
\Rightarrow \quad \langle h, M^2 h \rangle \leq \frac{\gamma}{\Pr(e)} \leq 2D \|h\|^2.
\]
Remark about a graph and its square