Coboundary Expansion & Locally Testable Codes

Recall:

"up" \( \delta : C^i \to C^{i+1} \)
\( \delta f(s) = \sum_{v \in s} f(s \setminus v) \)

"down" \( \partial : C^i \to C^{i-1} \)
\( \partial f(s) = \sum_{x(i) \in T \delta s} f(T) \)

Example: boundary of an edge \( \Theta \) a part \( \Theta \) a cycle

\[ \begin{array}{c}
\text{Lemma: } \delta \partial = 0 \quad \partial \delta = 0 \\
\text{proof: show true for each basis element.}
\end{array} \]

\( B_i = \text{Im } \delta_{i+1} \) boundary \( \bigcirc \)
\( Z_i = \ker \delta_i \) cycle

\( B^i = \text{Im } \delta_{i-1} \) co-boundary \( \bigcirc \)
\( Z^i = \ker \delta_i \) co-cycle

\( H^i = Z^i/B^i \quad H_i = Z_i/B_i \)

Example: in a graph, calculate \( H^0, H^1 \).

\( \delta_i : C^{i-1} \to C^i \) \( \text{Im } \delta_i = \) the constant \( \overline{0} \) or \( \Xi \) func.

\( \delta_0 : C^0 \to C^1 \) \( \ker \delta_0 = \{ f : \{x \} \mapsto \overline{0}, f(\alpha) = \chi(\overline{0}) \} \)
\( = \) constant on \( \alpha \) 's.
\[ \text{In } \delta_0 = \text{ cuts. } \quad \text{Ker } \delta^i_0 : \left\{ f : X(i) \rightarrow \mathbb{R}_f^3 : \forall u \mu : f(u) + f(v) = f(u') \right\} \]

\[ \Delta \text{'s sea even # of edges.} \]

Exercise: In the complete complex \( \mathbf{Z}^i = B^i \)

Proof: Suppose \( f : X(i) \rightarrow \mathbb{R}_f^3 \) s.t. \( \delta^i f = 0 \) \( (f \in \mathbf{B}^i) \).

We show that \( f \in \mathbf{B}^i = \text{Im } \delta_{i-1} \) by finding \( g : X(i-1) \rightarrow \mathbb{R}_f^3 \) s.t.

\[ \delta_{i-1} g = f. \]

Fix \( \omega_0 \) and set

\[ g(u_0, \ldots, u_i) \triangleq f(\omega_0, u_0, \ldots, u_i) \]

and

\[ g(u_0, u_{i+1}, \ldots, u_i) = 0 \quad u_j \neq u_0. \]

Check that \( f = \delta g : \)

\[ \forall \omega_0, u_0, \ldots, u_i : \delta g(u_0) = \sum_{j=0}^{i} g(u_0, u_{j+1}, \ldots, u_i) \]

\[ = \sum_{j=0}^{i} f(\omega_0, u_0, \ldots, u_j, \ldots, u_i) \]

since \( \delta^i f = 0 \)

so \( \delta^i f(u_0, u_0, \ldots, u_i) \)

Co-boundary Expansion:

\[ h^i = \text{sup } \frac{\text{wt}(\delta f)}{\text{dist}(f, B^i)} \]

for \( f \in \mathbf{Z}^i \backslash B^i \).
Thin [Gromov, LM]: Let $X = \Delta_n^{(d)}$.

$\forall 0 \leq i < d \quad \varphi^i > 1$

**Proof:** Fix $f \in C^i$, i.e. $f : X(i) \sim \Omega_{o,i}$. Suppose $w^t(\delta^f) < \epsilon$.

We will show that $\text{dist}(f, B^i) < \epsilon$.

If $f \in B^i$ this is obvious.

Otherwise, we find $g$ s.t. $f \neq dg$.

$\delta f < \epsilon \implies$

\[ \text{Prob} \left\{ \delta f(v_0 \cdots v_{i+1}) \neq 0 \right\} < \epsilon \]

\[ \text{Prob} \left\{ v_{i+1} \right\} < \epsilon \]

\[ \text{Fix } v_0 \text{ s.t. } \exists \]

Define $g(s) = \begin{cases} f(v_0 u s) & \text{if } v_0 \in s \\ 0 & \text{if } v_0 \notin s \end{cases}$

Check distance of $dg$ and $f$. 
\[ \delta g(S) = \sum_{x \in S} g(S \setminus x) \]

\[ v_0 \notin S = \sum_{x \in S} f(v_0 \cup S \setminus x) \]

\[ = f(S) \]

\[ \text{iff } \delta f(v_0 \cup S) = 0 \]

\[ \text{with } \text{prob } > 1 - \eta. \]

\[ \delta g(s) = f(s) \iff \delta f(v_0 \cup s) = 0 \]

\[ \delta g(v_0 \cup T) = g(T) + \sum_{x \in T} g(v_0 \cup T \setminus x) \]

\[ f(v_0 \cup T) \quad O \]

always true.

\[ \text{prob } \left( \frac{\delta g(S) = f(S)}{S} \right) > 1 - \eta. \]

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Comment: \( f \) was "corrected" to \( \delta g \).
\( \delta g \) "clearly" is in \( B \)
and \( \delta g \) is close to \( f \) (using the data
re \( \delta H \) is 0)

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**Locally Testable Code**

Linear space, defined by LDPC.

Test: associate a \( \nu t \) to each parity check.

"robustness of the
" test" = \( \frac{\nu t \text{ (rejecting checks)}}{\text{distance from code}} \)

Example: "linearity testing"

\[
\text{Code} = \{ f \in \{0,1\}^n \rightarrow \{0,1\}^n \mid f \text{ is } \mathbb{F}_2 \text{-linear} \}
\]

\[
= \{ f: \{0,1\}^n \rightarrow \{0,1\}^n \mid \forall x, y \in \mathbb{F}_2^n, f(x)+f(y)=f(x+y) \}
\]

\[ Hf = 0 \quad H = \left[ \begin{array}{c} z \\ \vdots \\ \delta f \end{array} \right] \]
Claim: The two sets are equivalent.

Proof: Let $\alpha_i = f(e_i)$.

clearly $f(\sum_x \alpha_i x_i) = \sum_x \alpha_i x_i = \langle \alpha, x \rangle$

Then [BLR]: if $\text{Prob} \left( f(x) \neq g(y) \right) > 1 - \varepsilon$

$\forall x, y$ then $\exists g$ linear s.t. $\text{Prob}(f = g) > 1 - 2\varepsilon$.

There are two very different proofs.

(a) Fourier analytic

(b) combinatorial "coboundary" style.

\[ f : \{\pm 1\} \rightarrow \{\pm 1\} \quad f = \sum \hat{f}(s) x_s \]

\[ \text{Prob} \left( f(x) f(y) \cdot f(x, y) = 1 \right) > 1 - \varepsilon \]

$\quad \forall x, y$ mult pointwise

$\Rightarrow \quad \text{E}_{x, y} f(x) f(y) f(x, y) > 1 - 2\varepsilon$
= \mathbb{E} \left( \sum_{x,y} \hat{p}(R,x,y) \right) \\
= \mathbb{E} \left( \sum_{x,y} \hat{p}(R,x,y) \right) \\
= \sum_{x,y} \hat{p}(R,x,y) \\
\Rightarrow \exists s \text{ s.t. } \hat{p}(s) > 1 - 2\varepsilon \\
\Rightarrow p > 1 - 2\varepsilon !

NP, CSPs; robustness and expansion

the landscape of solutions

Given $\phi$ a constraint system. E.g. a 3sat instance.

sat($\phi$) = $\{ a \in \{0,1\}^n : \phi(a) = \text{true} \}$

rej($\phi$) = frac of unhappy constraints.

hist($a$, sat($\phi$)) vs rej($a$).

\textbf{Def} Constraint Expander :
Generalizes several notions of PT, LTC, expansion:

1. If \( \phi \) is a graph, and each edge is \( \gamma \), this is edge expansion.

2. If \( \phi \) are constraints LDPC \( \rightarrow \) LTC

3. If \( \phi \) are cycle constraints on a S.C., this is cosystolic expansion (like coboundary expansion, but distinct if the cohomology is \( \neq 0 \)).

\[
\chi(\phi) = \min_{a \in \text{sat}(\phi)} \frac{\max_{a \in \phi} \text{dist}(a, \text{sat}(\phi))}{\text{dist}(a, \text{sat}(\phi))}
\]

\[\text{PCP theorem : [GP]}:\exists \gamma_0 \geq 0 \land \text{Alg from } (V_1, C_1) \text{ to } (V_2, C_2) \text{ s.t. } \gamma(C_2) > \gamma_0 \text{ and there is a bijection } \text{sat}(C_1) \leftrightarrow \text{sat}(C_2)\]
Corollary: it is $NP$-hard to decide if a given $\varphi$ has $val = 1$ or $val < 1 - \delta$.

Proof of Corollary:

Start with $NP$-hard $\varphi_1$, convert to $\varphi$, using expansion.

If alg can decide between $\varphi_1$ and $\varphi_0$ then

if $\varphi_1$ is sat $\implies$ $\varphi$ is sat
if $\varphi_1$ is unsat $\implies$ $\varphi$ isn't even $\forall_0$-sat

$(A \land \text{dist}(\varphi, \text{sat}(\varphi)) = 1 \implies$

\[
\frac{\text{prob}}{\frac{1}{2}} > \gamma
\]