Constructing Spectral HDX
(Kaufman - Oppenheim 2018)

We've seen Cayley graphs of groups & how they expand in the Abelian case. It is known that you cannot have expanders w bounded degree this way.

Today we meet a beautiful & different group theoretic construction, based on a group & its subgroups.

G - a group

A, B, C - subgroups

$AB \quad BC \quad AC$ - subgroups

$AB =$ group generated by A, B : $\{ w_1 \dots w_k \mid k \in \mathbb{N}, w_i \in A \cup B \}$

Warmup : Abelian groups

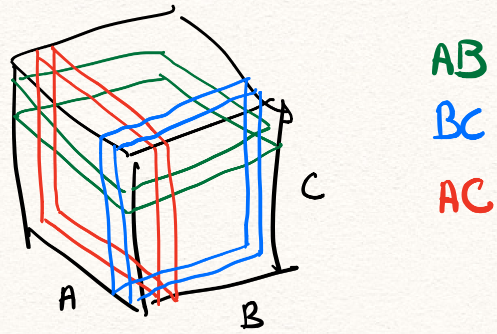
$A = \{(a,0,0) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$

$B = \{(0,b,0) \mid b \in \dots \}$

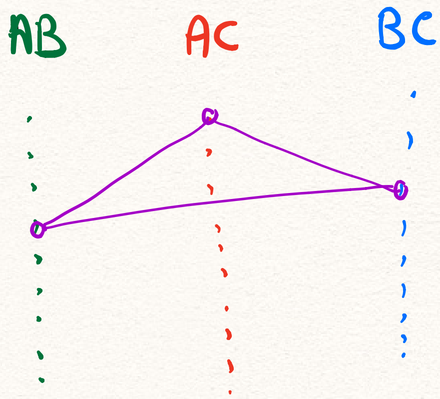
$C = \{(0,0,c) \mid c \in \dots \}$

$G = \{(a,b,c)\} \qquad AB = \{(a,b,0)\}$

The group G is covered by cosets of each subgroup



Our graph will be made of these cosets connecting iff they intersect



* every point in G is contained in exactly one coset of each color

$\Rightarrow$ makes a triangle

* every triangle corresponds to an intersection (needs proof)

* Symmetry : every pair of intersecting cosets have the same size intersection.

$\Leftrightarrow$ every edge is in the same # of triangles

vertices : 2-dim slices

edges : 1-dim bars : row or col or shaft

triangles : "0-dim" points

Graph : complete 3-partite graph ...

degree of each vertex is $2n$ edges , n^2 triangles

(not bounded degree)

The non-Abelian Case.

To get better degree bounds we want the cosets to be small wrt group size.

We will see a large (infite even) group generated by 3 tiny (finite) abelian groups.

(How can abelian groups generate a non-abelian group?)

$$A = \left\{ \begin{pmatrix} 1 & a & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} : a \in R \right\} \qquad R = \{ \alpha_1 t + \alpha_0 : \alpha_0, \alpha_1 \in \mathbb{F}_q \}$$

$$B = \left\{ \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & b \\ \cdot & \cdot & 1 \end{pmatrix} : b \in R \right\}$$

$$C = \left\{ \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ c & \cdot & 1 \end{pmatrix} : c \in R \right\}$$

<u>Claim:</u> A (or B or C) is abelian

Calc

<u>Claim:</u> $A \cap B = \{id\}$

What happens if we multiply $a \in A$ and $b \in B$?

$$\begin{pmatrix} 1 & a & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & b \\ \cdot & \cdot & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & ab \\ \cdot & 1 & b \\ \cdot & \cdot & 1 \end{pmatrix} \begin{matrix} \leftarrow \text{dg } 2 \\ \leftarrow \text{dg } 1 \\ \end{matrix}$$

$$\begin{pmatrix}1&\cdot&\cdot\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}\begin{pmatrix}1&a&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&1\end{pmatrix}=\begin{pmatrix}1&a&0\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}$$ so this group is not abelian...

<u>**Def**</u>: The group AB is called the <u>Heisenberg group</u>

<u>**Claim**</u>:

$$AB = \left\{\begin{pmatrix}1&a&\gamma\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix} : \begin{matrix}a,b\in R_1\\ \gamma\in R_2\end{matrix}\right\}\qquad \begin{matrix}R_1=\{\alpha_1 t+\alpha_0\}\\ R_2=\{\alpha_2 t^2+\alpha_1 t+\alpha_0\}\end{matrix}$$

proof ($\subseteq$ is immediate)

$$\begin{pmatrix}1&a&\gamma\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}\begin{pmatrix}1&a'&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&1\end{pmatrix}=\begin{pmatrix}1&a'+a&\gamma\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}$$ $\to$ can make a whatever we want

similarly $\to$ - - - b - -

$$\begin{pmatrix}1&a&\gamma\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}\cdot\begin{pmatrix}1&a'&\gamma'\\ \cdot&1&b'\\ \cdot&\cdot&1\end{pmatrix}=\begin{pmatrix}1&a'+a&\gamma+\gamma'+ab'\\ \cdot&1&b'+b\\ \cdot&\cdot&1\end{pmatrix}$$

Similarly **BC**, **CA** are copies of same group, but placed differently:

$$\underbrace{\begin{pmatrix}1&\cdot&\cdot\\ \cdot&1&b\\ \cdot&\cdot&1\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ c&\cdot&1\end{pmatrix}=\begin{pmatrix}1&\cdot&\cdot\\ bc&1&b\\ c&\cdot&1\end{pmatrix}}_{BC}\qquad \underbrace{\begin{pmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ c&\cdot&1\end{pmatrix}\begin{pmatrix}1&a&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&1\end{pmatrix}=\begin{pmatrix}1&a&\cdot\\ \cdot&1&\cdot\\ c&ac&1\end{pmatrix}}_{CA}$$

$|A| = q^2$

$|AB| = q^2\cdot q^2\cdot q^3 = q^7$

<u>**Claim**</u>: $G = \mathrm{span}(A, B, C) = \{ w_1 \ldots w_k \mid k\in\mathbb{N}\ \ w_i\in A\cup B\cup C\}$ is <u>infinite</u>

<u>"**Proof**"</u>:

$$\begin{pmatrix}1&a&b\\ &1&c\\ &&1\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ \cdot&d&1\end{pmatrix}=\begin{pmatrix}1&\overset{dg=3}{db+a}&b\\ \cdot&dc+1&c\\ \cdot&d&1\end{pmatrix}$$

Instead of working in an infinite setup, we restrict ourself to the ring of deg d polys (with truncation)

$$f(t)\cdot g(t) := \mathrm{dg}_d\big(f\cdot g\,(t)\big)$$

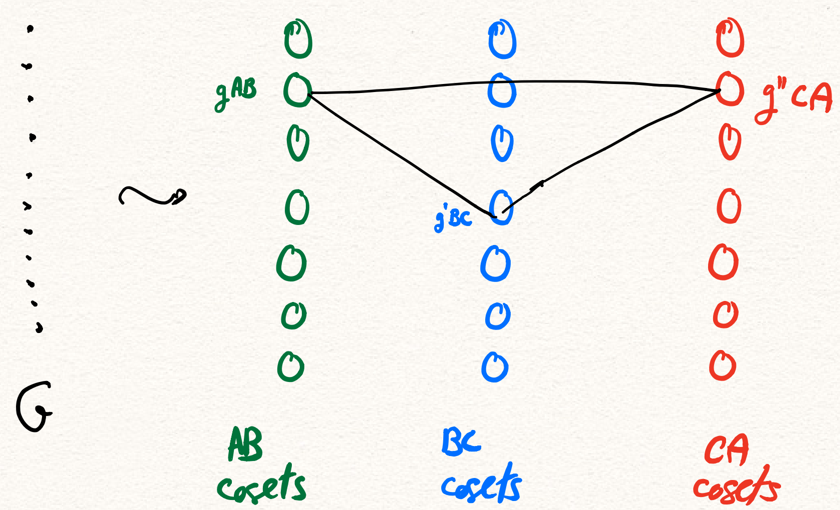
$+, \cdot$ associative, commutative, 1, .

$$|G| = \underbrace{q^d}_{\substack{\to\infty \\ \text{with } d\to\infty}} \qquad \underbrace{|AB| = q^7 \quad |A| = q^2}_{O(1)}.$$

What does the complex look like?

Cover G with cosets of AB, of BC, of CA.

For each type we have a color, and a vertex for each coset.



Place an edge if $gAB \cap g'BC$

We need to understand the intersections of the cosets.

Claim: (i) $AB \cap BC = B$; $BC \cap CA = C$; $CA \cap AB = A$

(ii) $B \cap C = \{id\}$

(iii) $A \cap BC = \{id\}$

Same as in abelian example

Proof. $BC \cap AB = \begin{pmatrix} 1 & \cdot & \cdot \\ * & 1 & * \\ * & \cdot & 1 \end{pmatrix} \cap \begin{pmatrix} 1 & * & * \\ \cdot & 1 & * \\ \cdot & \cdot & 1 \end{pmatrix} = B$

Claim: (a) if $gA \cap g'B \neq \phi$
then $|gA \cap g'B| = 1$

(b) if $gAB \cap g' BC \neq \phi$

then $gAB \cap g' BC = g'' B$

Proof: (a) if $x, y \in gA \cap g'B$

then $\exists\, \alpha \in A$ s.t. $x = y \cdot \alpha$

$x = ga \qquad y = ga' = ga \cdot \underbrace{(a^{-1}a')}_{\alpha}$

similarly $\exists\, \beta \in B$ s.t. $x = y\beta$

but $\alpha = \beta \in A \cap B = \{id\}$ so $x = y$.

$\Rightarrow |gA \cap g'B| = 1$.

(b) $\supseteq$: mult on left by $b \in B$ leaves us inside

$\subseteq$ if $h_1, h_2 \in g''AB$ then $\in AB$

$g\,BC$ $\quad$ $(\exists h$ s.t. $h_1 \cdot h = h_2)$ $\quad$ BC

$\Rightarrow h_1 \backslash h_2 \in B$

Claim : if (a) $g_1 AB \cap g_2 BC \neq \phi$

(b) $g_2 BC \cap g_3 CA \neq \phi$

(c) $g_1 AB \cap g_3 CA \neq \phi$

then $g_1 AB \cap \underbrace{g_2 BC \cap g_3 CA}_{gC} \neq \phi$

and has size 1.

gC vs $g' AB$:

check that $AB \cap C = \{id\}$, indeed

if gx, $gxy \in gAB \cap g'C$
$\rightarrow y \in AB \cap C = \{id\}$.

[intersections in "far away" cosets behave] like in the identity coset.

Conclusion: There is a 1-1 mapping between

(by construction) vertices : cosets of AB BC CA

(since any intersection is a coset) edges : cosets of A B C

(since any 3-wise intersection is a point) triangles : cosets of $\{id\}$, i.e. elements of G.

$\Longrightarrow$ We have a 2-dim complex $X(0), X(1), X(2)$

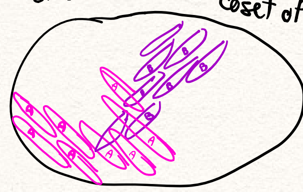
s.t. $|X(0)| = |G/AB| + |G/BC| + |G/CA|$

$$= 3 \cdot |G|/q^7 \xrightarrow[d\to\infty]{} \infty$$

$$|X(2)| = |G|$$

The link structure

one vertex = coset of AB

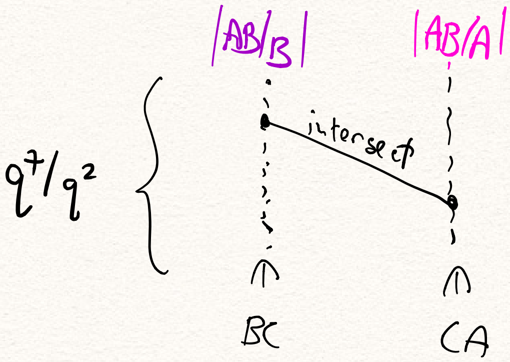


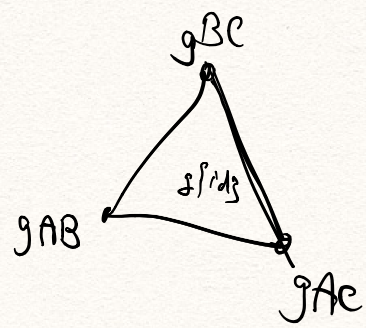
link of $x = gAB$

<u>Nbrs of x</u>: $\forall$ coset of A in AB we have 1 nbr in the cosets of CA

$\forall$ coset of B in AB - — — — - - BC

so locally (from the point of view of x) the picture is this





\# edges $= q^7$

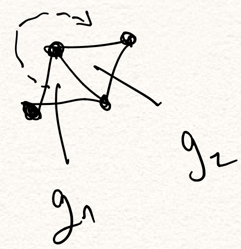
To understand the links we only need to understand the Heisenberg group (whose size is finite: q^7)

<u>fact</u>: $\lambda \le \frac{1}{\sqrt{q}}$.

We can use Garland's method (as per the thm of Oppenheim) to prove that the 1-skeleton too expands. For this, all we need is to establish

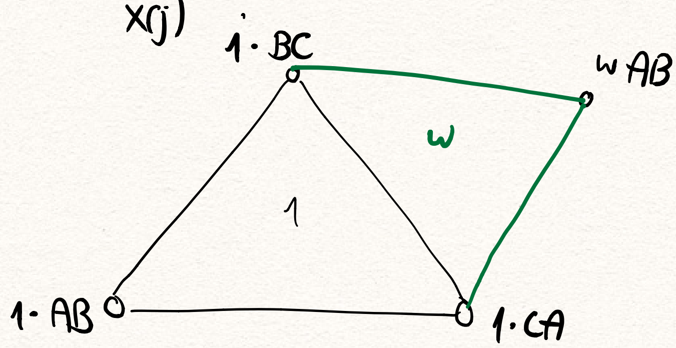
Connectedness

The group action on X:



⊛ Our group $G = \text{span}(A,B,C)$ acts on the complex, by mult, mapping simplices to simplices. (and paths to paths)

if $\underset{X(i)}{v} \subset \underset{X(j)}{S}$ then $\underset{X(i)}{w \cdot v} \subset \underset{X(j)}{w \cdot S}$



⊛ choose some $w \in C \setminus \{id\}$ and look at $x \curvearrowright wx$

$1 \cdot BC \curvearrowright w \cdot BC = 1 \cdot BC$ because $w \in C \subset BC$

$1\, CA \curvearrowright w \cdot CA = 1 \cdot CA$ "

$1\, AB \curvearrowright w \cdot AB \neq 1 \cdot AB$ (since $AB \cap C = \{id\}$)

Ex: find the stabilizer of a vertex, an edge

Question: $wAB \cap 1 \cdot CA = ?$ since $AB \cap CA \ni a$

$wAB \cap wCA \ni wa$

$wAB \cap CA$

Claim: The complex is "gallery connected", i.e. there is a $\Delta \frown \Delta \frown \Delta$ walk between any pair of Δ's.

Proof: enough to show that $1 \in G$ is connected to $g \in G$

write $g = w_1 \cdots w_k$ where $w_i \in A \cup B \cup C$.

we have just seen that 1 is connected to w_1

enough to see that $w_2 \cdots w_k$ is connected to 1

and shift all by w_1.

to find: $w_1 \sim\!\sim\!\sim w_1 \cdot (w_2 \cdots w_k)$

we find: $1 \sim\!\sim\!\sim (w_2 \cdots w_k)$

and then mult path by w_1.

we get $1 \frown w_1 \underset{\text{induction}}{\sim\!\sim\!\sim} w_1 \cdot (w_2 \cdots w_k)$. $\blacksquare$