Lecture 5

eigenvalues of Cayley graphs.

Let \( \Gamma \) be an abelian group, eg \( \mathbb{Z}^n \) or \( \mathbb{Z}/n\mathbb{Z} \).
Let \( S \subseteq \Gamma \) be a subset, and assume that if \( s \in S \) then \(-s \in S\).
\( \text{Cay}(\Gamma, S) \) is the graph whose vertices are \( \Gamma \) and edges are \( \{(a,b) : a-b \in S\} \).

This graph is 1-regular. Eigenvalues?

Examples:
\( \Gamma = \mathbb{Z}/n\mathbb{Z} \) \( S = \{ \pm 1 \} \) graph = cycle
\( \Gamma = \mathbb{Z}^n \) \( S = \{ e_1, -e_2 \} \) graph = Hamming cube.

Spectral decomposition: governed by characters.

**Def**: A character is a homomorphism \( \chi : \Gamma \to \mathbb{C} \setminus \{0\} \)

i.e. \( \chi(a) \chi(b) = \chi(a+b) \) \( \forall a,b \in \Gamma \).

- \( \chi(e) = 1 \)
- \( \chi \) is bounded: \( 1 = \chi(a+a+\ldots+a) = \chi(a) \chi(a) \ldots \chi(a) \Rightarrow |\chi(a)| = 1 \)
- \( \chi(a) \) is a root of unity

\[ \sum a \chi(a) = 0 \] unless \( a = 1 \).

**Proof**: \( \forall x, b \) s.t. \( x(b) = 1 \). \( \sum_a \chi(a) = \chi(b) \sum \chi(a+b) \ldots \)

**Def**: inner product: \( f, g : \Gamma \to \mathbb{C} \)

\[ <f, g> = \frac{1}{|\Gamma|} \sum_a f(a) \overline{g(a)}. \]

- \( \chi \chi \) and \( \chi \) are also characters (the set of characters form a group)
- \( <\chi, \chi> = 0 \)

\[ \chi(a) \overline{\chi(a)} = \chi(a) \overline{\chi(a)} \), either \( \chi \Psi = 1 \) or \( \sum = 0 \).

\( \chi \) the characters are orthogonal in \( \mathbb{C}^n \), so there are \( \leq |\Gamma| \) of them.
Theorem: if $\Gamma$ is finite, then $dim$ characters $= |\Gamma|$. 

proof. \[ \mathbb{Z}/\mathbb{Z}$ has $n$ characters \]
\[ \mathbb{Z} \times \mathbb{Z}$ has $n^2$ characters \]
\[ \mathbb{Z}/2^n \mathbb{Z}$ has $2^n$ characters \]

- look at $\chi_0 = e^{\frac{2\pi i}{2^n}} x \Rightarrow \chi_x(ab \mod 2) = \chi_x(a)\chi_x(b)$

$\chi_0(1)$ is different from all others.

(b) $\chi_1, \chi_2$ chars of $\mathbb{Z}/2^n \mathbb{Z}$ \Rightarrow $\chi_1(ab) = \chi_1(a)\chi_1(b)$

- if $\chi_i = \chi_k$, then $i = k$.

- if $i \neq k \exists a \neq 0$ s.t. $\chi_i(a) = \chi_k(a)$

so $\chi_j(a) = \chi_i(a)\chi_k(a) = \chi_{i+k}(a) \Rightarrow i = k$.

- every abelian group is $\cong (\mathbb{Z}/2^n \mathbb{Z})_n$.

Example: the Boolean Cube $\Gamma = (\mathbb{Z}/2^n \mathbb{Z})^n$

character of $\mathbb{Z}/2^n \mathbb{Z}$: $\chi \equiv 1$ $\Rightarrow$ $\chi_x = e^{\frac{2\pi i}{2^n}x} = (-1)^x$

$n$-fold product: $\chi_{x_1 \cdots x_n}(x) = \prod_{i=1}^{n} \chi_{x_i}(x_i)$

$\chi_{x_1 \cdots x_n}(i) = -1$.

$\chi_1(\mathbb{Z}/2^n \mathbb{Z}) \rightarrow \mathbb{C}$

$\chi = \prod_{i=1}^{n} \chi_{x_i}$

where $f_\chi = \sum_{i=1}^{2^n} f(x)\chi_i(x)$

The Boolean Fourier Transform

\[ f : \mathbb{Z}/2^n \mathbb{Z} \rightarrow \mathbb{C} \] can be written as

\[ f = \sum_{\chi} \hat{f}(\chi) \chi \]

where

\[ \hat{f}(\chi) = \langle f, \chi \rangle = \frac{1}{2^n} \sum_{x} f(x) \chi(x) \] characeter
Example: \( \mathbb{Z}/n\mathbb{Z} \) is the discrete Fourier transform.

Every \( f: (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C} \) can be written as

\[
f(\omega) = \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \hat{f}(r) e^{2\pi i \frac{r\omega}{n}}
\]

where \( \hat{f}(r) = \frac{1}{n} \sum_{k=0}^{n-1} f(k) e^{-2\pi i \frac{kr}{n}} \)

Remark:
(related to the group \([0,1)\) under addition mod 1,
periodic functions on \( \mathbb{R} \) are functions \( f: [0,1) \to \mathbb{C} \)
and it turns out that the characters of \([0,1)\)
are \( x_k = e^{2\pi i k x} \) and \( f = \sum_{k=0}^{n-1} f(k) e^{2\pi i k x} \)
is the Fourier series of \( f \).

Back to Cayley graphs...

Lemma: \( P \) finite abelian group, \( \chi \) a character.
\( S \) a symmetric set.
Let \( M \) be the Markov operator.
Let \( x \in \mathbb{C}^P \) be such that \( x_a = \chi(a) \).
Then \( x \) is an e.v. \( u \) w.r.t. \( \chi \).
real vs complex ? \( \chi \) is real (why?)
so, \( \text{re}(\lambda) \) is also an e.v.

Remarkable: e_vectors can
de fixed ind. of \( S \), only \( \lambda \) depend
on \( S \).

Eigenvalues of the cycle \( (\text{exercise}) \)

Eigenvalues of the cube:

\[ \chi_r - \text{what is } \lambda_r \]

\[ \frac{1}{|S|} \sum_{s \in S} \chi_r(s) = \frac{1}{r} \sum_j \chi_r(e_j) = \frac{1}{r} \sum_j (1)^j \]

\[ (-1)^r \sum_{i=1}^{r} \]