Lecture 5

eigenvalues of Cayley graphs.

Let Γ be an abelian group, eg $\mathbb{Z}_2^n$ or $\mathbb{Z}/n\mathbb{Z}$.

Let $S \subset \Gamma$ be a subset, and assume that if $s \in S$ then $-s \in S$.

$Cay(\Gamma, S)$ is the graph whose vertices are Γ

edges are $\{(a,b) : a-b \in S\}$

This graph is $|S|$-regular. Eigenvalues?

Examples: $\Gamma = \mathbb{Z}/n\mathbb{Z}$ $\quad S = \{\pm 1\}$ $\quad$ graph = cycle

$\Gamma = \mathbb{Z}_2^n$ $\quad S = \{e_1, \ldots e_n\}$ $\quad$ graph = Hamming cube.

Spectral decomposition: governed by characters.

<u>Def</u> A character is a homomorphism $\chi : \Gamma \to \mathbb{C} \setminus \{0\}$

i.e. $\chi(a)\chi(b) = \chi(a+b) \quad \forall a, b \in \Gamma$.

- $\chi(0) = 1$
- χ is bounded : $1 = \chi(a + \cancel{a+} \ldots + a) = \chi(a)\chi(a)\ldots\chi(a)$

  $\Rightarrow |\chi(a)| = 1$

  $\Rightarrow \chi(a)$ is a root of unity

- $\sum_a \chi(a) = 0$ unless $\chi \equiv 1$.

  proof: fix b s.t. $\chi(b) \neq 1$. $\sum_a \chi(a) = \chi(b) \sum_a \chi(a+b) \ldots$

<u>Def</u>: inner product: $f, g : \Gamma \to \mathbb{C}$ $\quad \langle f, g \rangle = \frac{1}{|\Gamma|} \sum_a f(a)\overline{g(a)}$.

- $\chi_1\chi_2$ and $\overline{\chi}$ are also characters (the set of characters form a group)
- $\langle \chi, \psi \rangle = 0$

$\chi(a)\overline{\psi(a)} = \chi\overline{\psi}\,(a)$. either $\chi\overline{\psi} \equiv 1$ or $\Sigma .. = 0$.

$\rightarrow$ the characters are orthog in $\mathbb{C}^\Gamma$, so there are $\leq |\Gamma|$ of them.

Theorem: if Γ is finite abelian then # characters $= |\Gamma|$.

proof. (a) $\mathbb{Z}/n\mathbb{Z}$ has n characters

(b) $\Gamma_1 \times \Gamma_2$ have characters corr to $\{\chi_i \psi_j\}$

(a) — Look at $\chi_r(x) = e^{2\pi i \frac{r}{n} x}$ $\chi_r(a+b \mod n) = \chi_r(a)\chi_r(b)$

$\chi_r(1)$ is diff $\forall r$.

(b) χ_1, χ_2 chars of Γ_1, Γ_2 $\longrightarrow (ab) \to \chi_1(a)\chi_2(b)$ is a character

χ_1', χ_2' $\chi_{12}(ab) = \nearrow$

if $\chi_{ij} = \chi_{kl}$ then: if $i \neq k$ $\exists a$ s.t. $\chi_i(a) \neq \chi_k(a)$

so $\chi_{ij}(a0) = \chi_i(a) \neq \chi_k(a) = \chi_{kl}(a0) \Rightarrow i = k$.

(every abelian group is $\cong (\mathbb{Z}/n_i\mathbb{Z})_i$

Example: <u>the Boolean Cube</u> $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$

character of $\mathbb{Z}/2\mathbb{Z}$: $\chi_{r=0} \equiv 1$ $\chi_{r=1} = e^{2\pi i \frac{1}{2} \cdot x} = (-1)^x$

$\chi_1(0) = 1, \ \chi_1(1) = -1.$

n-fold product: $\chi_{r_1 \dots r_n}(x) = \prod_{i=1}^n \chi_{r_i}(x_i)$

$\chi : (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow \mathbb{C}$

$$= \prod_{i=1}^n (-1)^{r_i x_i} = (-1)^{r_1 x_1 + \dots + r_n x_n}$$

The Boolean Fourier transform

$\forall f : \{0,1\}^n \to \mathbb{C}$ cab be writen as $f = \sum \hat{f}(S) \chi_S$

character

where $\hat{f}(S) := \langle f, \chi_r \rangle = \frac{1}{2^n} \sum_x f(x) \chi_r(x)$

Example : $\mathbb{Z}/n\mathbb{Z}$ the discrete Fourier transform

Every $f : (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ can be written as

$$f(x) = \sum_{r \in \{0,\dots,n-1\}} \hat{f}(r)\, e^{2\pi i \frac{r}{n} x}$$

where $$\hat{f}(r) = \frac{1}{n} \sum_{a=0}^{n-1} f(a)\, e^{2\pi i \frac{a}{r}}$$

remark:

(related to the group $[0,1)$ w addition mod 1, periodic functions on $\mathbb{R}$ are functions $f: [0,1) \to \mathbb{C}$ and it turns out that the characters of $[0,1)$ are $\chi_r(x) = e^{2\pi i r x}$ and $f = \sum_{r \in \mathbb{Z}} \hat{f}(r)\, e^{2\pi i r x}$ is the Fourier series of f. and $\hat{f}(r) = \int_0^1 f(x)\chi_r(x)\,dx$)

Back to Cayley graphs...

Lemma: Γ finite ablelian group. χ a character.
S a symmetric set.
Let M be the Markov operator
Let $x \in \mathbb{C}^{\Gamma}$ be such that $x_a = \chi(a)$.
Then x is an e.v. w ev $\lambda = \frac{1}{|S|}\sum_s \chi(s)$.

real vs complex? λ is real (why?)

So, $re(X)$ is also an e.v.

Remarkable: evectors can be fixed ind. of S. only λ depend on S.

Eigenvalues of the cycle (exercise)

Eigenvalues of the cube:

χ_r — what is λ_r

$$\frac{1}{|S|}\sum_{s\in S} \chi_r(s) = \frac{1}{n}\sum_j \chi_r(e_j) = \frac{1}{n}\sum_j (-1)^{r_j}$$

$(-1)^{\sum r_i x_i}$

$$= \frac{1}{n}(n-r + -r)$$

$$= 1 - \frac{2r}{n}.$$