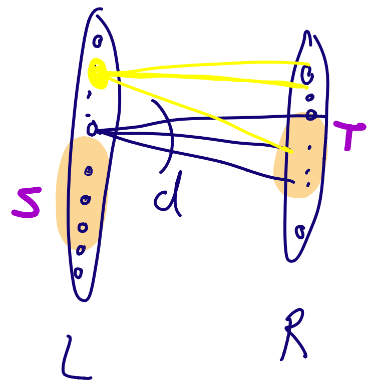
Existence of Magical Graphs

Lemma: $\exists n_0\ \forall n \geq n_0$, $d \geq 32$, $m \geq \frac{3}{4}n$ $\quad \exists\ (n,m,d)$ graph s.t.

(1) $\forall S \subset L \quad |S| \leq \frac{n}{3d} \quad |\underbrace{\Gamma(S)}_{\{u \in R \mid u \text{ is a nbr of some } v \in S\}}| > \frac{5}{8} d \cdot |S|$

(2) $\forall S \subset L \quad \frac{n}{3d} < |S| \leq \frac{n}{2} \quad |\Gamma(S)| > |S|$



Proof: (sketch) We will select G at random and show that $\text{Prob}(G \text{ is not a magical graph}) < 1$.

– $\forall v \in V$ randomly select $u \in R$ d times ind.

Prove that (2) holds whp.

$\forall S \subset L \ \frac{n}{3d} < |S| \leq \frac{n}{2} \qquad \forall |T| = |S| \qquad X_{S,T}$ indicates $\Gamma(S) \subset T$.

if $\sum X_{S,T} = 0 \longrightarrow$ (2) holds

$$\text{Prob}_G \left(\sum_{S,T} X_{S,T} > 0 \right) \leq \sum_{S,T} \underbrace{\text{Prob}_G \left(X_{S,T} \neq 0 \right)}_{\left(\frac{t}{m}\right)^{sd}}$$

$$= \sum_{s=1}^{\frac{n}{3d}} \binom{n}{s} \binom{m}{s} \left(\frac{t}{m}\right)^{s \cdot d}$$

$$\left(\frac{ne}{s}\right)^s \left(\frac{me}{s}\right)^s \left(\frac{s}{m}\right)^{s \cdot d}$$

* lin. prog. are nearly optimal for lin. trans

why?

* disconnecting fewer than s means low rank.

simplifying assumption: $m = n$

$m = \beta n$

$\beta \geq \frac{3}{4}$

$$\left(\frac{ne}{s}\right)^{2s} \left(\frac{s}{n}\right)^{sd}$$

$$\left(\underbrace{\left(\frac{ne}{s}\right)^2 \left(\frac{s}{n}\right)^d}_{\to 0} \right)^s$$

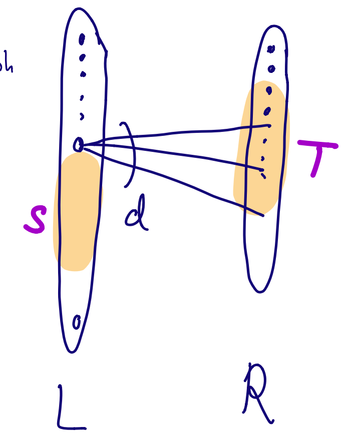
$$\frac{s^{d-2}}{n^{d-2}}$$

$$\left(\frac{ne}{s}\right)^s \left(\frac{\beta n e}{s}\right)^s$$

$$\left(\left(\frac{ne}{s}\right)^2\right)^s \beta^s \left(\frac{s}{\beta n}\right)^{sd}$$

$$\left[\left(\frac{ne}{s}\right)^2 \left(\frac{s}{\beta n}\right)^d \cdot \beta \right]^s$$

(Non-explicit) Existence of "Magical Graphs"



Lemma: $\exists n_0 > 0$ s.t. $\forall n > n_0$ $d \geq 32$ $m \geq \frac{3}{4} n$ $\exists (n,m,d)$ graph

(1) $\forall S \subset L$ $|S| \leq \frac{n}{10d}$ $|\Gamma(S)| \geq \frac{5}{8} \cdot d|S|$

(2) $\forall S \subset L$ $\frac{n}{10d} < |S| \leq \frac{n}{2}$ $|\Gamma(S)| > |S|$

Proof Sketch: "prob. method"

Every $v \in L$ chooses $u \in R$ d times unif. ind.

prove Prob_G ((2) is satisfied) $> \frac{1}{2}$.

Fix $T, S \subset L$ $\frac{n}{10d} < |S| \leq \frac{n}{2}$, $|T| = |S|$. $X_{S,T}$ - indicates $\Gamma(S) \subseteq T$

observe: if $\sum_{S,T} X_{S,T} = 0$ then (2) is satisfied.

S_0, T_0 : $\Pr_G (X_{S_0,T_0} \neq 0) = \left(\frac{t}{m}\right)^{sd}$

$$\text{Prob}\Big(\sum_{S,T} X_{S,T} \neq 0 \Big) \leq \sum_{S,T} \text{Prob}\big(X_{S,T} \neq 0\big) = \sum_{s = \frac{n}{10d}+1}^{\frac{n}{2}} \overbrace{\binom{n}{s}}^{\text{choice of } S} \overbrace{\binom{m}{s}}^{\text{choice of } T} \left(\frac{s}{m}\right)^{sd}$$

$$\leq \left(\frac{ne}{s}\right)^s \left(\frac{me}{s}\right)^s \left(\frac{s}{m}\right)^{sd}$$

$$= \Big[\underbrace{\frac{ne}{s} \cdot \frac{me}{s} \cdot \left(\frac{s}{m}\right)^d} \Big]^s$$

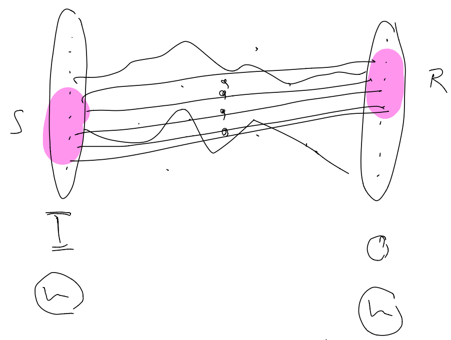
$$\leq \sum_{s}^{\frac{n}{2}} \binom{n}{s}\binom{m}{s}\left(\frac{s}{n}\right)^{sd}$$ Pretend $n = m$

$$\Big[\underbrace{\left(\frac{ne}{s}\right)^{2\cancel{s}} \left(\frac{s}{n}\right)^{\cancel{s}d}}_{\substack{\text{as small} \\ \text{as needed } (< \frac{1}{10}) \\ \text{by increasing } d}} \Big]^s$$

$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$

$\leq \frac{1}{9}$

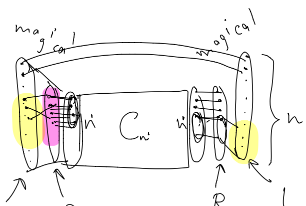
Using magical graphs $\longrightarrow$ superconcentrators



Recursive construction

$n < n_0$ output complete bip. graph

Given sup-con $n \longrightarrow$ sup-con $\frac{4}{3}n$



$n = \frac{4}{3}m$ $\quad m = n'$

Prove that C_{n+1} is a sup-con.
Fix S, T size k

⊛ case $k \leq \frac{n}{2}$
use property ② + Halls thm ...

⊛ $k > \frac{n}{2}$
use L–L matching edges

Sparsity:

$E_1 \leq V_1 \cdot D$

$$E = E_1 + E_2 \leq D \cdot V_1 + D \cdot V_2 = D \cdot (V)$$

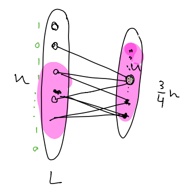
$+ V_2$
$+ E_2$

$|E_2| \leq \overbrace{(4d+4)}^{D} \cdot |V_2|$

~~Expanders~~ $\longrightarrow$ Error Correcting Codes

Magical Graph (1)

$C \subseteq \underset{\mathbb{F}_2}{\{0,1\}^n}$ lin. subspace $\mathbb{F}_2$



$C = \left\{ w \in \{0,1\}^L \;\middle|\; \forall u \in R \quad \sum_{v \sim u} w(v) = 0 \bmod 2 \right\}$

$\dim C \geq \frac{n}{4}$ r. rate $\geq \frac{1}{4}$

distance ?? $w_1, w_2 \in C$

$w = w_1 + w_2$ is a non-zero $\in C$

$wt(w) = dist(w_1, w_2)$

Goal: $\forall w \in C \setminus \{0\}$ $wt(S) \geq \underset{\frac{1}{10d}}{\delta} \cdot n$.

Assume for contradiction $|S| < \frac{n}{10d}$

$S = \{v \mid w(v) \neq 0\}$

$\Gamma(S) > |S| \cdot \frac{5}{8}d \xrightarrow{\text{prove}}$ S has a unique neighb nbr

(otherwise $|\Gamma(S)| \leq \frac{d}{2} \cdot |S|$)

The unique nbr is a violated constraint!

Success Amplification RP

Suppose $A(x, \underline{r})$ k random bits

is a rand. alg for L w

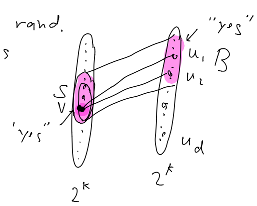
$\forall x \in L$ $\underset{r}{\text{Prob}}(A(x,r) = \text{yes}) = 1$

$\forall x \notin L$ $\underset{r}{\text{Prob}}(A(x,r) = \text{yes}) \leq \frac{1}{16} \longrightarrow \varepsilon$

Goal: error ε no more random bits.

$\forall x$ let B = set of bad rand. strings

$|B| \leq \frac{1}{16} \cdot 2^k$



if $|S| > \frac{n}{10d}$

choose some $S_1 \subset S$ $|S_1| = \frac{n}{10d}$

$\Gamma(S_1) \geq \frac{5}{8}d \cdot \frac{n}{10d} = \frac{n}{16}$

So $\Gamma(S) \not\subset B$. $\varepsilon \leq \frac{1}{10d}$.