Combinatorial def of Expansion

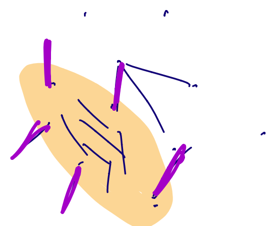
$G=(V,E)$
d-regular

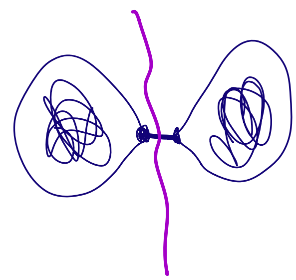
$$h(G) = \min_{\substack{\phi \neq S \subset V \\ |S| \leq \frac{|V|}{2}}} \frac{|E(S,\bar{S})|}{|S| \cdot d}$$

"combinatorial edge expansion"

"cheeger constant of the graph"

$S \subset V$ $\qquad \underline{\partial S} = E(S, \underbrace{\bar{S}}_{V \setminus S}) = \{ \{u,v\} \in E,\ u \in S,\ v \in \bar{S} \}$





A graph G is a (d,ε)-expander if it is d-regular & $h(G) > \varepsilon$.

A family of graphs is a (d,ε)-exp fam. if

$G_1, G_2, G_3, \ldots$ $\quad G_i$ is on n_i vertices $\underline{n_1 < n_2 < \ldots}$

$\forall i \quad h(G_i) > \varepsilon \quad \& \quad G_i$ is d-regular.

- $\{G_i\}$ is mildly explicit : $\exists$ alg, s.t. given 1^i generates G_i in poly-time.
- $\{G_i\}$ is very explicit : ($\exists$ alg) given $\underline{i, \underset{[n_i]}{v}, \underset{[d]}{k}}$ generates the name of the

k-th nbr of v in poly-time.
(i.e. poly($\log i + \log n_i + \log d$)

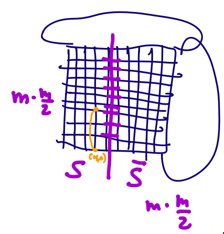
Examples:

- Margulis 1980's $\quad V = \mathbb{Z}_m \times \mathbb{Z}_m$

$$V \ni (x,y) \sim \begin{array}{ll} (x+y,\ y) & (x-y,\ y) \\ (x+y+1,\ y) & (x-y+1,\ y) \\ & \\ (x,\ y+x) & (x,\ y-x) \\ (x,\ y+x+1) & (x,\ y-x+1) \end{array} \qquad \begin{array}{l} d = 8 \\ \\ |V| = m^2 \end{array}$$

EXPANDER $\quad$ + mod m

- ~~Grid~~ graph $\quad \mathbb{Z}_m \times \mathbb{Z}_m \qquad (x,y) \sim (x, y\pm 1),\ (x\pm 1, y)$
Torus

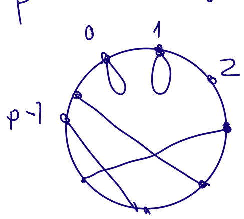


$$h(\text{Torus}) \leq h(S) = \frac{E(S,\bar{S})}{4\cdot|S|} \approx \frac{m}{4\cdot\frac{m^2}{2}} = \frac{1}{2m} \to 0$$

NOT an EXPANDER

- LPS '88 $\quad$ Ramanujan Expanders

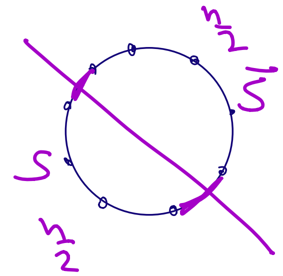
$V = \mathbb{F}_p$ finite field $\qquad x \sim x+1,\ x-1,\ x^{-1} \in \mathbb{F}_p$



$d = 3 \qquad$ mod p

$2^{-1} = 3 \pmod 5$

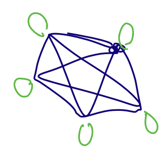
- Cycle graph C_n



NOT an Expander

$$h(S) = \frac{E(S,\bar{S})}{2\cdot|S|} = \frac{2}{n} \xrightarrow[n\to\infty]{} 0$$

- Complete graph



$d = n-1$

<u>not sparse</u>

$$S \subset V \qquad h(S) = \frac{E(S,\bar{S})}{|S|\cdot n} = \frac{\cancel{|S|}\cdot|\bar{S}|}{\cancel{|S|}\cdot n} \in \left(\tfrac{1}{2}, 1\right)$$

- Boolean hypercube

  Hamming cube.

$V = \{0,1\}^k \leftarrow \mathbb{F}_2^k$ vector space

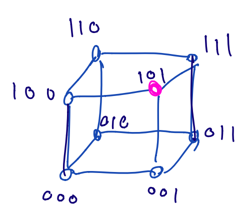
$x \sim x + e_i \mod 2 \quad i = 1..k$

$$e_i = (0-0\,1\,0-0)$$

$\uparrow$ i-th

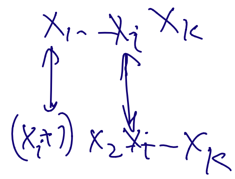
$$d = k = \log_2 |V|.$$

$h(G)$



Best (sparsest) cut is "coordinate cut"

$$S_i = \{x \in \{0,1\}^k \mid x_i = 0\} \qquad \bar{S}_i = \{ \dots x_i = 1\}$$



every x has 1 edge across the coord. cut.

$$h(S_i) = \frac{|S_i| \cdot 1}{|S_i| \cdot k} = \frac{2^{k-1}}{2^{k-1} \cdot k} = \frac{1}{k}$$

Not an expander, k (degree) is unbounded from above.

$\frac{1}{k}$ in unbouned from below.

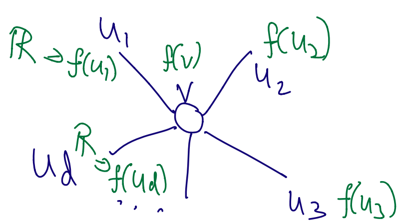
Spectral / Algebraic Graph Theory

$G \longleftrightarrow$ Adj Matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} v_1 \; v_2 \; \cdots \; v_n \\ \left[\begin{array}{ccccccccc} & & & & & & & & \\ 0 & 1 & 0 & 1 & & 0 & 0 & 1 & 1 \; 1 \\ & & & \boxed{} & & & & & \\ & & & & & & & & \end{array}\right] \end{array}$$

#edges between v_i & v_j

G d-reg normalized adj $M = \frac{1}{d} \cdot A$



$\{ f : V \to \cancel{\{0,1\}} \}$ $\mathbb{R}$

$$g = \underbrace{A}_{n \times n} f \qquad n = |V|$$

$$g(v) = f(u_1) + f(u_2) + \dots + f(u_d) = (Af)(v)$$

<u>Fact:</u> Every sym. matrix has n orthogonal eigenvectors, with <u>real</u> eigvalues.

$f \in \mathbb{R}^V$ is eig. vec iff $A\vec{f} = \lambda \cdot \vec{f}$ $\quad \lambda \in \mathbb{R}$.

Given G, let $y_1, \dots, y_n \in \mathbb{R}^V$ be e.vectors

$\lambda_1 \dots \lambda_n$ corr e.values.

By convention $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

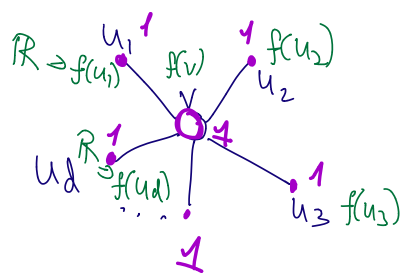
Let M be the norm. adj of d-reg G. $\lambda_1 \geq \dots \geq \lambda_n$ be its ev's.

<u>Claim 1:</u> $\lambda_1 = 1$.

Let $z = 1111\dots1$ $\quad Mz$

$Mz = \overline{1} = 1 \cdot z$.

So, $\lambda_1 \geq 1$



Suppose y is ev w $\lambda > 1$. $My = \lambda y$.

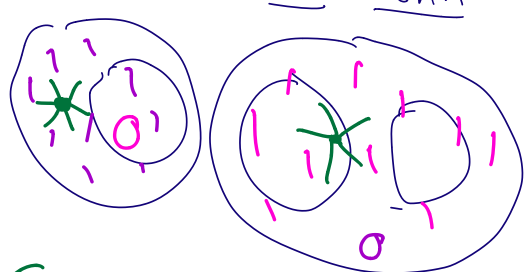
supp. v is such $y_v = \max_i y_i$

$\lambda \cdot y(v) = (My)(v) = \text{avg of nbrs} \le y(v)$

$\rightarrow \lambda \le 1$.

Claim 2 G is connected iff $\lambda_2 < \lambda_1 = 1$.

Proof: G is not conn $\longrightarrow \lambda_2 = \lambda_1$



$z_1 = (1 \quad\quad 0) \qquad Mz_1 = 1 \cdot z_1$

$z_2 = (0 \quad\quad 1) \qquad Mz_2 = 1 \cdot z_2$

$z_1 + z_2 = (1\ 1 \ldots 1)$

Suppose $\lambda_1 = \lambda_2$, prove G is disconnected.

Let x be an eigenvector. $x \neq \alpha \mathbb{1}$

$x_{max} = x(v) \qquad x_{min} = x(u) \qquad u \neq v$ because $x \neq$ const.

Assume v is a vertex s.t. $x(v) = x_{max}$
u $x(u) = x_{min}$

Then u, v are in distict components.