Lecture 4

Expander Mixing Lemma EML

G d-reg graph $G=(V,E)$

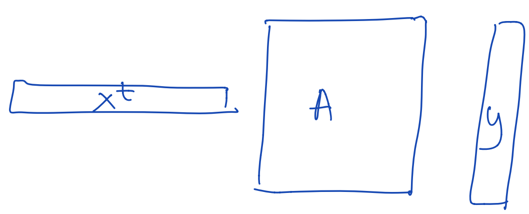
A adj matrix $A(i,j) =$ # edges bet i & j

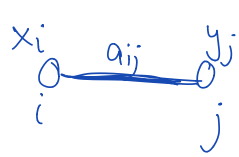
M normalized, i.e. $M = \frac{1}{d} \cdot A$.

A is also a quadratic form

$$x^t A y = \qquad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n)$$

$$= \sum_{i,j=1}^{n} A_{ij} x_i y_j$$





$$\vec{x} = \overrightarrow{\mathbb{1}_S} \qquad \vec{y} = \overrightarrow{\mathbb{1}_T} = \begin{cases} 1 & i \in T \\ 0 & i \notin T \end{cases}$$

For $S, T \subset V$

$$\mathbb{1}_S^t \, A \, \mathbb{1}_T = \sum_{i,j=1}^{n} A_{ij} \, \mathbb{1}_S(i) \, \mathbb{1}_T(j)$$

$$= \sum_{i \in S} \sum_{j \in T} A_{ij}$$

$$= |E(S,T)|$$

$$\text{''}$$

edges from S to T

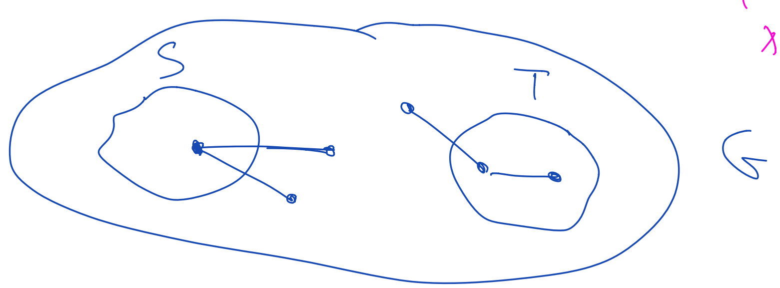
$i,j \in S \cap T \qquad A_{ij} + A_{ji}$

<u>Expander Mixing Lemma</u>:

Let G be a d reg graph, let $d = \lambda_1 \geq \lambda_2 \geq \dots$ be the ev of its adj mat.

$$\left| \, |E(S,T)| - \frac{|S| \cdot |T| \cdot d}{n} \right| \leq \cancel{\lambda_2} \; \lambda \sqrt{|S| \cdot |T|}$$

(where $\lambda = \max(|\lambda_2|, |\lambda_n|)$)



$|S|\, d \cdot \frac{|T|}{n}$ - expected number of edges from S to T if each v had chosen its nbrs randomly.

<u>Proof</u>: $x = \mathbb{1}_S \quad y = \mathbb{1}_T \quad x, y \in \mathbb{R}^V$

Let $u_1 \ldots u_n$ be orthonormal eigenvectors of A.

(Recall $\mathbb{R}^V$ has an inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$

$\|x\| = \left(\sum x_i^2\right)^{1/2} = \langle x, x \rangle^{\frac{1}{2}}$)

$u_1 = \frac{1}{\sqrt{n}} \cdot \vec{1} \qquad \left(\|u_1\| = \sqrt{\sum_{i=1}^n \left(\frac{1}{\sqrt{n}}\right)^2} = 1 \right).$

$x = \sum_{i=1}^n \alpha_i u_i$ for some coefs $\alpha_1, \ldots \alpha_n$

$y = \sum_{j=1}^n \beta_j u_j$ " " $\beta_1 \ldots \beta_n$

$$\langle x, u_j \rangle = \left\langle \sum_{i=1}^n \alpha_i u_i, u_j \right\rangle$$

$$= \sum_{i=1}^n \alpha_i \langle u_i, u_j \rangle = \underline{\alpha_j}$$

$$\alpha_1 = \langle x, u_1 \rangle = \sum_{i=1}^{n} x(i)\, u_1(i) = \sum_{i \in S} u_1(i) = \frac{|S|}{\sqrt{n}}$$

$$\beta_1 = \frac{|T|}{\sqrt{n}}$$

$$\overrightarrow{Ay} = A\left(\sum_{j=1}^{n} \beta_j u_j\right) = \sum_{j=1}^{n} \beta_j \underbrace{A u_j}_{\lambda_j u_j} = \sum \beta_j \lambda_j u_j$$

$$\langle \vec{x}, \overrightarrow{Ay} \rangle = \left\langle \sum_{i=1}^{n} \alpha_i u_i , \sum_{j=1}^{n} \beta_j \lambda_j u_j \right\rangle$$

$$= \sum_{i,j} \alpha_i \beta_j \lambda_j \langle u_i , u_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i \lambda_i \cdot 1$$

$$\langle x, Ay \rangle = x^t A y$$

$$= \sum_i x_i \underline{(Ay)}_i = \sum_i x_i \sum_j A_{ij} y_j = \sum A_{ij} x_i y_j$$

$$= x^t A y$$

$$= |E(S,T)|$$

$$|E(S,T)| = \alpha_1 \beta_1 \cdot d + \underbrace{\sum_{i>1} \lambda_i \alpha_i \beta_i}$$

$$= \frac{|S|}{\sqrt{n}} \cdot \frac{|T|}{\sqrt{n}} \cdot d + $$

$$\left| |E(S,T)| - |S| \cdot |T| \cdot \frac{d}{n} \right| = \underbrace{\left| \sum_{i>1} \lambda_i \alpha_i \beta_i \right|}$$

$$\leq \lambda \sum_{i>1} |\alpha_i \beta_i|$$

where $\lambda = \max(|\lambda_n|, |\lambda_2|)$.

$$\sum_{i>1}^{n} |\alpha_i||\beta_i| \underset{\substack{\uparrow \\ \text{Cauchy Schwartz}}}{\leq} \overset{\|x\|}{\left(\sum_{i>1} \alpha_i^2\right)^{1/2}} \cdot \left(\sum_{i>1} \beta_i^2\right)^{1/2} \leq \underbrace{\|1_S^\perp\|} \cdot \|1_T^\perp\|$$

[Recall $\langle x, x \rangle = \langle \sum \alpha_i u_i, \sum \alpha_i u_i \rangle = \sum \alpha_i^2$, so $\|x\| = (\sum \alpha_i^2)^{1/2}$]

$$\| 1_S \| = \underbrace{\left(\sum_{i=1}^{n} 1_S(i)^2 \right)}^{1/2} = \sqrt{S}$$

Conclusion: $\left| |E(S,T)| - \frac{d |S| \cdot |T|}{n} \right| \leq \lambda \cdot \sqrt{|S| \cdot |T|}$

∎

Applications

- Independent Set in G $S \subset V$ if $|E(S,S)| = 0$.

Def: A graph G is an (n,d,λ) graph if it is d-reg & $\max(\lambda_2, |\lambda_n|) \leq \lambda$.

Claim: max ind. set in an (n,d,λ)-graph is $\leq \frac{\lambda}{d} \cdot n$.

Proof: Suppose $S \subset V$ is s.t. $E(S,S) = 0$.

$$\left| \cancel{E(S,S)}_{0} - \frac{d|S| \cdot |S|}{n} \right| \leq \lambda \sqrt{|S| \cdot |S|}$$

$$\frac{d \cdot |S|}{n} \leq \lambda$$

$$|S| \leq \frac{\lambda \cdot n}{d}.$$

Corollary: $\chi(G) \geq \frac{d}{\lambda}$

• Expanders have small diameter $O(\log n)$

Enough to prove that if $|S| \le \frac{n}{2}$

$E(S,\bar{S}) > \varepsilon \cdot |S|$.

$$E(S,S) \le \underbrace{\frac{d\,|S|^2}{n} + \lambda \cdot |S|}$$

$$\frac{E(S,\bar{S})}{|S|} \ge \underbrace{d \cdot |S| - \left(\qquad \right)}_{S}$$

$$= d - \frac{d|S|}{n} - \lambda = d\left(1 - \frac{|S|}{n}\right) - \lambda$$

$$\ge \frac{d}{2} - \lambda$$

If $s \le \frac{n}{3}$ this is $\ge \frac{2}{3}d - \lambda$

If $\underline{\lambda < \frac{d}{2}}$ $\quad \frac{E(S,\bar{S})}{|S|} \ge \varepsilon = \frac{d}{2} - \lambda$