worksheet- class work

(from Spielman's book :)

1. Orthogonal eigenvectors. Let $\boldsymbol{M}$ be a symmetric matrix, and let $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ be vectors so that

$$\boldsymbol{M}\boldsymbol{\psi} = \mu\boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{M}\boldsymbol{\phi} = \nu\boldsymbol{\phi}.$$

Prove that if $\mu \neq \nu$ then $\boldsymbol{\psi}$ must be orthogonal to $\boldsymbol{\phi}$. Note that your proof should exploit the symmetry of $\boldsymbol{M}$, as this statement is false otherwise. e.g. for $M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

2. Invariance under permutations.

Let $\boldsymbol{\Pi}$ be a permutation matrix. That is, there is a permutation $\pi : V \to V$ so that

$$\boldsymbol{\Pi}(u, v) = \begin{cases} 1 & \text{if } u = \pi(v), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that if

$$\boldsymbol{M}\boldsymbol{\psi} = \lambda\boldsymbol{\psi},$$

then

$$\left(\boldsymbol{\Pi}\boldsymbol{M}\boldsymbol{\Pi}^T\right)(\boldsymbol{\Pi}\boldsymbol{\psi}) = \lambda(\boldsymbol{\Pi}\boldsymbol{\psi}).$$

That is, permuting the coordinates of the matrix merely permutes the coordinates of the eigenvectors, and does not change the eigenvalues.

3. Invariance under rotations.

Let $\boldsymbol{Q}$ be an orthogonal matrix. That is, a matrix such that $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$. Prove that if

$$\boldsymbol{M}\boldsymbol{\psi} = \lambda\boldsymbol{\psi},$$

then

$$\left(\boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^T\right)(\boldsymbol{Q}\boldsymbol{\psi}) = \lambda(\boldsymbol{Q}\boldsymbol{\psi}).$$

4. Similar Matrices.

A matrix $\boldsymbol{M}$ is similar to a matrix $\boldsymbol{B}$ if there is a non-singular matrix $\boldsymbol{X}$ such that $\boldsymbol{X}^{-1}\boldsymbol{M}\boldsymbol{X} = \boldsymbol{B}$. Prove that similar matrices have the same eigenvalues.

5. Spectral decomposition.

Let $\boldsymbol{M}$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and let $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n$ be a corresponding set of orthonormal column eigenvectors. Let $\boldsymbol{\Psi}$ be the orthogonal matrix whose ith column is $\boldsymbol{\psi}_i$. Prove that

$$\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{\Psi} = \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal. Conclude that

$$\boldsymbol{M} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^T = \sum_{i \in V} \lambda_i \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T.$$

6. Traces.

Recall that the trace of a matrix $\boldsymbol{A}$, written $\operatorname{Tr}(\boldsymbol{A})$, is the sum of the diagonal entries of $\boldsymbol{A}$. Prove that for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{Tr}(\boldsymbol{B}\boldsymbol{A}).$$

Note that the matrices **do not** need to be square for this to be true. They can be rectangular matrices of dimensions $n \times m$ and $m \times n$.

Use this fact and the previous exercise to prove that

$$\operatorname{Tr}(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$. You are probably familiar with this fact about the trace, or it may have been the definition you were given. This is why I want you to remember how to prove it.

7. The trace method.

Let A be the adj matrix of a d-reg graph.

$d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let $\lambda = \max(\lambda_2, -\lambda_n)$, so $\underset{\forall i > 1}{\lambda \geq |\lambda_i|}$

- show that $Tr(A^2) = n \cdot d$ (more generally $Tr(A^k) = ?$)
- show that $Tr(A^2) = \sum_{i=1}^{n} \lambda_i^2$

conclude that $\lambda \geq \sqrt{d} - o_n(1)$

[Alon-Boppana showed $\lambda \geq 2\sqrt{d-1}$, and a Ramanujan graph is a d-reg graph with $\lambda = 2\sqrt{d-1}$. Such were constructed by LPS '88]