The Spectral Theorem

Let M be a sym. $\overset{n \times n}{\vee}$ matrix. Then there is a basis $v_1 .. v_n \in \mathbb{R}^n$ for $\mathbb{R}^n$ s.t. $Mv_i = \lambda_i \cdot v_i$ for all $i = 1..n$.

Proof: the Rayleigh Quotient $\dfrac{x^t M x}{x^t x}$

Since RQ is homog. we assume $\|x\| = 1$.

$S = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ is a compact set.

thus, the max. value of $x^t M x$ is attained for some $x \in S$.

if the thm is true, we can write $x = \sum \alpha_i v_i$ for evecs $v_1 .. v_n$

$$\underline{x^t M x} = \langle \sum \alpha_i v_i , \sum \alpha_i \lambda_i v_i \rangle = \underbrace{\sum \alpha_i^2 \lambda_i}_{\uparrow} \leq \underset{\substack{\uparrow \\ \text{largest}}}{\lambda_1}$$

$\Rightarrow$ the value of the Rayleigh Quotient is λ_1.

Look at $f(x_1 ... x_n) = \dfrac{x^t M x}{x^t x}$ $\quad \nabla x = 0$ at maximum.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \right)$$

$$\nabla \underbrace{x^t x}_{\substack{\sum x_i^2 \\ \uparrow}} = (2x_1, 2x_2, 2x_3, \ldots) = 2x$$

$$\nabla x^t M x = 2 M x$$

$$\nabla \qquad \left(\frac{f}{g}\right)' = \pm \frac{f g' - g f'}{g^2} \quad +$$

$$\nabla \frac{x^t M x}{x^t x} = \frac{2 Mx \cdot x^t x - 2x \cdot x^t M x}{(x^t x)^2} = 0$$

$$2Mx \cdot x^t x = 2x \cdot x^t M x$$

$$Mx = x \cdot \underbrace{\frac{x^t M x}{x^t x}}_{\in \mathbb{R}} \qquad (*)$$

x, the maximizer, sats $(*)$ so it is an eigenvector!

we already saw that $\frac{x^t M x}{x^t x} = \lambda$ is the ~~...~~ largest e.v.

Lets call this maximizer v_1.

v_1 $\qquad v_1^{\perp} = \{ y \in \mathbb{R}^n \mid \langle y, v_1 \rangle = 0 \}$

$y \in v_1^{\perp}$, $\quad \langle v_1, My \rangle \underset{\substack{\uparrow \\ M \text{ is sym.}}}{=} \langle M v_1, y \rangle = \langle \lambda v_1, y \rangle = \lambda \langle v_1, y \rangle$

$$\lambda_1 = \max_{\substack{x \neq 0 \\ \|x\| = 1}} \frac{x^t M x}{x^t x}$$

$$\lambda_2 = \max_{\substack{x \neq 0 \\ \|x\| = 1 \\ x \in v_1^{\perp}}} \frac{x^t M x}{x^t x}$$ let v_2 be a maximizer.

(v_1 denotes the maximizer from the prev step)

$\vdots$

λ_n

We keep going and get a sequence of eigen vectors $v_1, v_2, \dots, v_n$

$\square$

Theorem 2.0.1 (Courant-Fischer Theorem)**.** *Let $\boldsymbol{M}$ be a symmetric matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then,*

$$\mu_k = \boxed{\max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{\boldsymbol{x} \in S \\ \boldsymbol{x} \neq \boldsymbol{0}}} \frac{\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}}^{\,*} = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(T)=n-k+1}} \max_{\substack{\boldsymbol{x} \in T \\ \boldsymbol{x} \neq \boldsymbol{0}}} \frac{\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}},$$

where the maximization and minimization are over subspaces S and T of $\mathbb{R}^n$.

Proof: We know M has eigenvecs $v_1, \ldots, v_n$

choose $S = \operatorname{span}(v_1, \ldots, v_k)$.

$$\min_{x \in S} \frac{x^t M x}{x^t x} = \frac{\sum \alpha_i^2 \mu_i}{\sum \alpha_i^2} = \mu_k \qquad x = \sum_{i=1}^{k} \alpha_i v_i$$

$\Rightarrow \quad * \geq \mu_k$

We next show that $\displaystyle\min_{\substack{x \in S \\ x \neq 0}} \frac{x^t M x}{x^t x} \leq \mu_k$ for all S (k-dim spaces).

Let $T = \operatorname{span}(v_k, v_{k+1}, \ldots, v_n)$ $\dim T = n-k+1$

$\dim T \cap S \geq 1$

$$\min_{\substack{x \in S \\ x \neq 0}} \frac{x^t M x}{x^t x} \leq \min_{\substack{x \in S \cap T \\ x \neq 0}} \frac{x^t M x}{x^t x} \leq \max_{x \in T} \frac{x^t M x}{x^t x} = \mu_k$$

$\square$

Graphs & their matrices

The graph Laplacian $L = \underbrace{D}_{\substack{\text{degrees}\\ \text{on}\\ \text{diagonal}}} - \underbrace{A}_{\text{adjacency}}$

$d \cdot I$ for a d-reg graph

__Claim:__ $x^t L x = \sum_{\{ij\} \in E} (x_i - x_j)^2$

__Proof:__ $\sum_{ij \in E} (x_i - x_j)^2 = d \cdot \sum_{i \in V} x_i^2 - \sum_{\substack{i \neq j \\ ij \in E}} x_i x_j$

$= d \cdot x^t I x - \underbrace{x^t A x}_{\sum_{i,j} A_{ij} x_i x_j}$

$= x^t (dI - A) x = x^t L x.$ $\square$

__Eigenvalues of L:__ (focus on d-reg graphs).

if v is an eigenvec of A $\quad Av = \underbrace{\lambda}v$

$\Rightarrow$ v - - - - - L

$dI \cdot v = d \cdot v$

$Lv = (dI - A)v = d \cdot v - \lambda \cdot v = \underbrace{(d-\lambda)} \cdot v$

if A has e.v. $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

L $\quad 0 = d - \lambda_1 \leq d - \lambda_2 \leq \dots \leq d - \lambda_n$

Def: A sym matrix is called positive semi-definite if all ev's are non-negative. PSD

$\Rightarrow$ Laplacian is PSD.

Cheeger - Alon - Milman Inequality

Let G be a d-reg graph, let $1 = \lambda_1 \geq \lambda_2 \geq \dots$ be the normalized eigen vals.

$$\frac{1-\lambda_2}{2} \leq \underset{\substack{\uparrow \\ \text{edge expansion}}}{h} \leq \sqrt{2(1-\lambda_2)}$$

$E(S,\bar{S}) = \{ e = \{u,v\} \mid u \in S,\ v \notin S \}$

$$h = \text{"the edge expansion"} = \min_{\substack{\emptyset \neq S \subset V \\ |S| \leq \frac{n}{2}}} \frac{\overset{=}{|\partial S|}}{|S|\, d}$$

Cheeger constant

isoperimetric constant

$$\phi = \text{sparsity} = \min_{\emptyset \subsetneq S \subsetneq V} \frac{|\partial S|}{|S| \cdot |\bar{S}|} \cdot \frac{n}{d} \leq \min_{S} \frac{|\partial S|}{|S| \cdot d} \cdot \overset{2}{\cancel{\frac{n}{|\bar{S}|}}} = 2h.$$

$$\frac{G}{\text{complete}} = \frac{\dfrac{E_G(S,\bar{S})}{\frac{1}{2} d n}}{\dfrac{E_K(S,\bar{S})}{\frac{1}{2} n \cdot n}}$$

$\frac{1}{2} d n$ — $\# |E(G)|$

$\frac{1}{2} n \cdot n$ — $\# |E(K)|$

We will see $1-\lambda_2 \leq \phi \leq 2h$

$\uparrow$ immediate since $|\bar{S}| \geq n/2$.

We saw $\sum_{\{i,j\}\in E} (x_i - x_j)^2 = x^t L x$

choose $x = \mathbb{1}_S - s\mathbb{1}$ for $S \subset V$ $\quad x^t L x = |E(S,\bar{S})|$.

where $s = \frac{|S|}{|V|} = \frac{|S|}{n}$ $\qquad x(i) = \begin{cases} 1-s & i \in S \\ -s & i \notin S \end{cases}$

$$x^t x = \sum_{i=1}^{n} x_i^2 = \sum_{i\in S} (1-s)^2 + \sum_{i\notin S} (-s)^2$$

$$= n\cdot s\cdot(1-s)^2 + n(1-s)\cdot s^2 = n\cdot s\cdot(1-s)\cdot\left[1-\cancel{s} + s\right] = ns(1-s).$$

$$ns(1-s) = |S|\cdot\frac{|\bar{S}|}{n}$$

Write $\hat{L} = \frac{1}{d}\cdot L = I - \frac{1}{d}\cdot A$.

$$1-\hat{\lambda}_2 = \lambda_2(\hat{L}) = \min_{\substack{x\neq 0 \\ x\perp \vec{1} \\ x\in\mathbb{R}^n}} \frac{x^t \hat{L} x}{x^t x} \leq \min_{\substack{\phi\neq S\subsetneq V \\ x = \mathbb{1}_S - \frac{|S|}{n}\cdot\mathbb{1}}} \frac{E(S,\bar{S})\cdot\frac{1}{d}}{|S|\cdot|\bar{S}|\cdot\frac{1}{n}} = \phi.$$

$\hat{\lambda}_2 = d^{-1}\cdot\lambda_2(A)$

relaxation

natural combin question

second smallest $\qquad$ second largest

$$\mu_{n-1}(\hat{L}) = 1-\lambda_2(\hat{A})$$