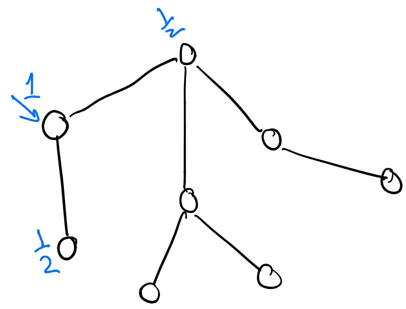
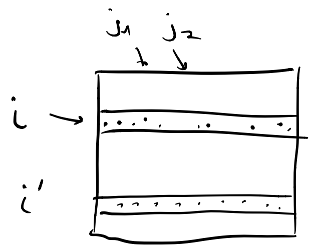
Random Walks on Graphs



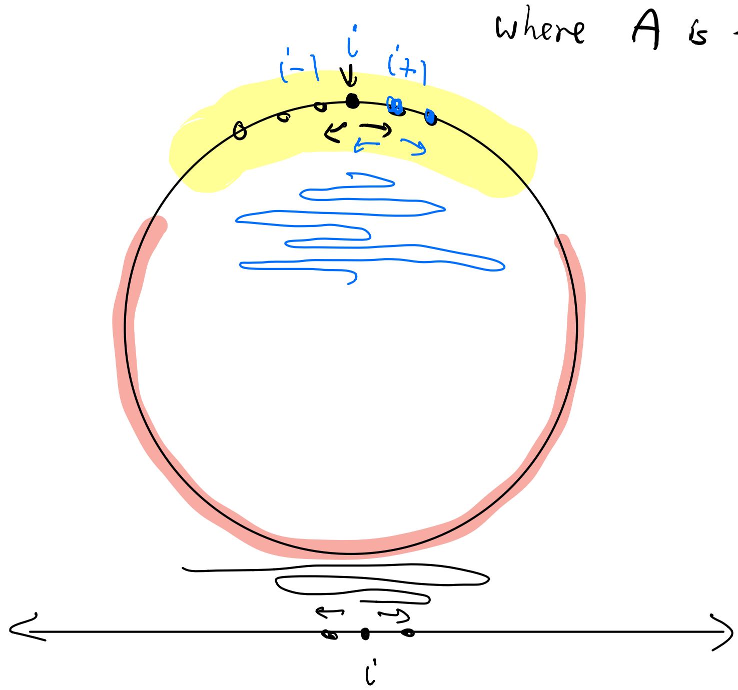
M - normalized adjacency matrix

$$M_{ij} = \text{Prob}(j \mid i) \quad \text{"transition prob. matrix"}$$

$= \text{probab. of landing at } j \text{ cond. on starting at } i$



Given a d-reg graph G, the norm. adj. matrix $M = \frac{1}{d} \cdot A$ where A is the adj. matrix.



C_n

Let $p \in \mathbb{R}^n$ be a prob. distr. over V $(|V| = n)$.

$p = (p_1, p_2, \dots, p_n)$ $\sum_{i=1}^{n} p_i = 1$ $p_i \geq 0$.

Suppose we choose a vertex $v \sim p$
and then choose <u>a random nbr</u> of v.

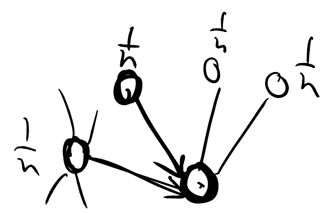
Let q be the new prob. distr.

$q = Mp$.

$$q_i = \sum \underbrace{\underset{P}{\mathrm{Prob}(j)}}_{p_j} \cdot \underbrace{\mathrm{Prob}(i \mid j)}_{M_{ji} = M_{ij}} = \sum M_{ij} p_j = (Mp)_i$$

Let u be the uniform distribution $u = (u_1, \dots, u_n) = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) = \frac{1}{n} \cdot \vec{1}$

$Mu = u$



$$d \cdot \frac{1}{n} \cdot \frac{1}{d} = \frac{1}{n}$$

Suppose $p \neq u$ is a prob. dist.

Mp $Mp \overset{?}{=} p$ cannot happen if G is connected

(because $\lambda_2 < \lambda_1$, and $p \notin sp(u)$
yet $Mp = p$ means $\lambda_2 = 1$)

conclusion: u is the <u>unique</u> stationary distr. if G is connected.

Question: How quickly (if at all) does a RW approach u?
in the worst case starting distr.

stat. distance between p, q is $\max_{A \subset V} |P(A) - Q(A)| = dist_{TV}(p, q)$

$$= \underbrace{\sum_{i=1}^{n} |p_i - q_i|}\cdot \frac{1}{2} = \frac{1}{2} \cdot \|p-q\|_1$$

(Recall : $\|z\|_1 = \sum_{i=1}^{n} |z_i|$

$$\|z\|_2 = \left(\sum_{i=1}^{n} z_i^2 \right)^{1/2}$$

$$\underline{\|z\|_1} = \sum_{i=1}^{n} |z_i| \cdot 1 \underset{\text{Cauchy-Schwartz.}}{\leq} \|z\|_2 \cdot \|1\|_2 = \sqrt{n} \cdot \underline{\|z\|_2}$$)

A RW on an expander is <u>rapidly mixing</u>

<u>Def</u>: A RW is a stochastic process defining a seq. of vertices $(X_0, X_1, X_2, \ldots, X_t, \ldots)$

where X_0 is a vertex chosen accord. init. distr.

X_{i+1} is chosen by a uniform nbr of X_i.

<u>Thm</u>: $\| M^t p - u \|_1 \leq \sqrt{n} \cdot \alpha^t$

$(0.9)^{(\log n) \cdot 100} = \frac{1}{poly(n)}$

where M is the norm adj mat. of a d-reg graph G with $\max(|\lambda_2|, |\lambda_n|) = \alpha$.

<u>Remark</u>: $M^t p$ is the distr after t steps of RW from p.

$$p \to \underset{\substack{\| \\ q}}{(Mp)} \qquad Mq = M(Mp) = M^2 p \quad \ldots \quad M^t p$$

Proof:

$$p = \underline{\underline{u}} + \underline{p-u}$$

$p-u \perp u$ $\qquad \langle p-u, u\rangle = \sum_{i=1}^{n} (p_i - u_i)\cdot u_i$

$$= \frac{1}{n}\Big(\underbrace{\sum p_i}_{1} - \underbrace{\sum u_i}_{1}\Big) = 0$$

let $y_1 \dots y_n$ be a basis of eigenvectors, corr to eigenvalues $|\lambda_1| \geq \lambda_2 \geq \dots$

$$p = \sum \alpha_i y_i = \hat{\alpha}_1 y_1 + \Big(\sum_{i>1} \overset{\lambda_i}{\alpha_i}\, y_i\Big)$$

wlog $y_1 = u$

$y_i \perp u$

$$= u + (p-u)$$

$$M^t p = M^t\big(u + (p-u)\big) = M^t u + M^t (p-u)$$

$$= u + \dots$$

$$u + \sum_{i>1} \alpha_i \cdot \lambda_i \cdot y_i$$

$$\|Mp - u\|_2 = \Big\|\sum_{i>1} \alpha_i \lambda_i y_i\Big\|_2 \leq \alpha \cdot \Big\|\sum_{i>1} \alpha_i y_i\Big\|_2 \leq \alpha \cdot \|p-u\|_2$$

$\max_{i>1} |\lambda_i| = \alpha$.

$$\|M^t p - u\|_2 = \|M^t (p-u)\| \leq \alpha^t \underbrace{\|p-u\|}_{\leq \|p\| \leq 1} \leq \alpha^t$$

$\|p\|_2^2 = \|p-u\|_2^2 + \|u\|_2^2$

$\langle p, p\rangle = \langle (p-u) + u, (p-u) + u\rangle = \langle p-u, p-u\rangle + \langle u, u\rangle$

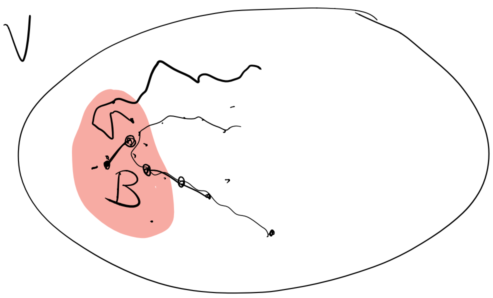
So $\|p-u\| \leq \|p\|$.

To see $\|p\| \leq 1$ use convexity: since $\sum p_i = 1$ $\|p\|$ is max when all mass is on p_1.

By CS ineq: $\|M^t p - u\|_1 \leq \sqrt{n} \cdot \alpha^t$. $\square$

Random walks immitate independent Sampling

V



$\sim \left(\frac{|B|}{n}\right)^t$

$v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_t$

$\frac{|B|}{n}$

$E(S,T)$

$B \qquad B$

Theorem: Let G be a (n, d, α)-graph

vertices   d-reg $\max(|\lambda_2|, |\lambda_n|)$.

Let $B \subset V$ $\quad \frac{|B|}{n} = \beta$.

$$\Pr_{v_1 \ldots v_t}\left[\forall i \quad v_i \in B\right] \leq (\alpha+\beta)^t$$

RW from uniform starting distr.

Proof:

M

P_B — projection onto B

$$\begin{array}{cc} & \begin{array}{cc} B & \bar{B} \end{array} \\ \begin{array}{c} B \\ \bar{B} \end{array} & \begin{pmatrix} \ddots & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{pmatrix} z \end{pmatrix} = \begin{pmatrix} z_B \\ 0 \end{pmatrix}$$

Let p be a prob. dist.

$p' = P_B p = ?$ $\quad \begin{array}{ccc} \bar{B} & & B \end{array}$ $(p_1 \ldots \ldots p_n)$

$p' = (0\,0\,0\,0\; p_{k+1} \ldots p_n)$

$$\sum_{i=1}^{n} (p')_i = \operatorname{Prob}_{i \sim p}(i \in B)$$

$$= p(B).$$

$$\|P_B u\|_1 = \sum_{i=1}^{n} (P_B u)_i = \sum_{i \in B} \frac{1}{n} + \sum_{i \notin B} 0 = \frac{|B|}{n}.$$

$$\|P_B M P_B u\|_1 = \operatorname{prob}(v_1 \in B \text{ and } v_2 \in B)$$

$\vdots$

$$\|(P_B M)^t P_B u\|_1 = \operatorname{prob}(\forall i \; v_i \in B)$$

$$\underline{(\alpha + \beta)^t}$$