# Week 5 - class work

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This week we will deepen our understanding of past material and explore new material through a series of guided exercises. Please work in groups, preferably through overleaf, and submit the written classwork one submission per group. We will decide on the submission date depending on the progress today and tomorrow.

This file can be found on Overleaf at: `https://www.overleaf.com/read/mqbqfvkpcxxj`

## 1 Efficient error reduction in probabilistic algorithms:

A language $L \in RP$ if there exists a polynomial randomized algorithm $A$ satisfying

1. If $x \in L$, then for every random string $r \in \{0,1\}^k$, $A(x,r) = 1$.
2. If $x \notin L$, then the probability over $r \in \{0,1\}^k$, that $A(x,r) = 1$ is at most $\beta$.

We would like to reduce the error probability of the algorithm while saving on random bits.

Observe that by repeating $A$ independently $t$ times, the error probability reduces to $\beta^t$, i.e. if $x \notin L$ then

$$\Pr_{r_1,\ldots,r_t}[A(x,r_1) = 1, A(x,r_2) = 1, \ldots, A(x,r_t) = 1] < \beta^t.$$

The number of random bits used in this way is $tk$. We next try to achieve a similar behaviour with fewer random bits. Let $G$ be an $(n,d,\lambda)$-graph, with $V = \{0,1\}^k$, and $\lambda < 1/2$. We then run the following algorithm $A'$:

- Choose $v_1 \in V$ uniformly at random.
- Perform a random walk $v_1, \ldots, v_t$.
- Accept iff for all $i \in [t]$: $A(x, v_i) = 1$.

**Goal:** Prove that for all $x \notin L$

$$\Pr[A'(x) = 1] < (\beta + \lambda)^t.$$

1. Let $B \subset \{0,1\}^k = V$ be the set of random strings causing the algorithm to accept. Express the probability that the algorithm accepts after one step (i.e. that a random walk of one step ($t = 2$) does not leave $B$).
2. Consider the projection operator $P$ defined by a $|V| \times |V|$ matrix defined as $\begin{cases} P_{v,v'} = 1 & v = v' \in B \\ P_{v,v'} = 0 & \text{otherwise} \end{cases}$.

   For a vector $f \in \mathbb{R}^{|V|}$ on $V$, describe the vector $Pf$.

3. Prove that the probability that a random walk of $t$ steps does not leave $B$ is equal to $\left\|(PM)^t Pu\right\|_1$, where the vector $u = (1/|V|, \dots, 1/|V|)$ denotes the uniform distribution, and $M \in \mathbb{R}^{V \times V}$ is the normalized adjacency matrix of $G$.

4. Prove that $\left\|(PM)^t Pu\right\|_1 \le \sqrt{|V|} \left\|(PM)^t Pu\right\|_2$.

5. Prove that for every vector $v \in \mathbb{R}^{|V|}$ we have $\|PMPv\|_2 \le (\beta + \lambda) \|v\|_2$. Use the following steps.
   - Prove that $\|Pv\|_2 \le \|v\|_2$. Conclude that we may assume without loss of generality that $v$ is supposted only on the coordinates from $B$, i.e. $Pv = v$.
   - Assume without loss of generality that $\sum_i v_i = 1$. (explain why)
   - Write $v = u + z$, where $u$ is the uniform vector and $z$ is some vector such that $\sum_i z_i = 0$. Then $PMPv = PMv = PMu + PMz$
   - Prove that $\|PMu\|_2 = \sqrt{\beta/n} \le \beta \|v\|_2$.
   - Prove that $\|PMz\|_2 \le \|Mz\|_2 \le \lambda \|z\|_2 \le \lambda \|v\|_2$.
   - Conclude that $\|PMPv\|_2 \le \|PMu\|_2 + \|PMz\|_2 \le (\beta + \lambda) \|v\|_2$.

6. Conclude that for every vector $v \in \mathbb{R}^{|V|}$ we have $\left\|(PM)^t Pv\right\|_2 \le (\beta + \lambda)^t \|v\|_2$.

7. Use item 4 to prove that $\left\|(PM)^t Pu\right\|_1 \le \sqrt{|V|} \left\|(PM)^t Pu\right\|_2 \le \sqrt{|V|}(\beta + \lambda)^t \|u\|_2 = (\beta + \lambda)^t$.

8. Use item 3 to conclude the goal.

## 2 Weighted irregular graphs

Suppose $G$ is a weighted not-necessarily-regular undirected graph. A random walk on $G$ moves from a vertex $v$ by selecting a random edge touching $v$ with probability *proportional to its weight*, and moves to the corresponding neighbor.

- Describe the transition matrix of this random walk.
- Suppose $G$ is an unweighted connected not-necessarily-regular graph. The stationary distribution of $G$ is a distribution on the vertices that remains unchanged after one step in the random walk. Write a precise description of the stationary distribution as a function of the vertex-degrees.
- Assume that $G$ is a weighted connected graph. Describe the stationary distribution for $G$.

## 3 Graph Products

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. (For some of the items here you may assume that $G$ is $d_1$-regular and $H$ is $d_2$-regular although this is not important). Consider two independent random processes: one process that is a random walk on $G$ and one process that is a random walk on $H$. Let $(x, y)$ be the current joint state of the two random walks, i.e. $x \in V_G$ and $y \in V_H$.

1. Describe the matrix $M_{G \times H}$ that is the Markov transition matrix of this random process, as a function of the random walk matrices $M_G$ and $M_H$ of the graphs $G, H$ respectively.

2. Describe the graph on vertices $V_G \times V_H$ whose random walk corresponds to $M_{G\times H}$. This graph is called $G \times H$.

3. * Suppose $G, H$ are weighted, connected, not-necessarily-regular, graphs. What is the stationary distribution of this walk as a function of the stationary distributions of $G, H$?

4. Give a description of the graph $G^{\times n} = G \times \cdots \times G$ ($n$ times).

5. * The $\varepsilon$-noisy hypercube graph has vertices $\{0,1\}^n$. The edge between vertices $x, y \in \{0,1\}^n$ has weight $(\varepsilon)^{\mathrm{dist}(x,y)}(1-\varepsilon)^{n-\mathrm{dist}(x,y)}$ where $\mathrm{dist}(x,y)$ is the number of coordinates on which $x, y$ differ. Describe a graph $G_\varepsilon$ on two vertices $\{0,1\}$ such that the $n$-fold product of this graph, is the noisy hypercube with parameter $\varepsilon$.

6. Suppose that $v_1, \ldots, v_n$ are eigenvectors of $G$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose $v'_1, \ldots, v'_m$ are eigenvectors of $H$ with eigenvalues $\lambda'_1, \ldots, \lambda'_m$. What are the eigenvectors and eigenvalues of $G \times H$ ? (prove your claim).

7. What are the eigenvectors and eigenvalues of your graph $G_\varepsilon$, and of $G_\varepsilon^{\times n}$?