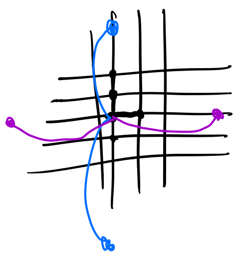
Gabber - Galil - Margulis

Expander Graph

1973 Margulis

'79 Gabber - Galil

$\mathbb{Z}^2$



$S(x,y) = (x+y, y)$ $\quad S^{-1}(x,y) = (x-y, y)$

$T(x,y) = (x, y+x)$ $\quad T^{-1}(x,y) = (x, y-x)$

$$G = \left(V = \mathbb{Z}^2 \setminus \{(0,0)\},\ E = \left\{ (x,y) \to S(x,y),\ S^{-1}(x,y),\ T(x,y),\ T^{-1}(x,y) : (x,y) \neq (0,0) \right\} \right)$$

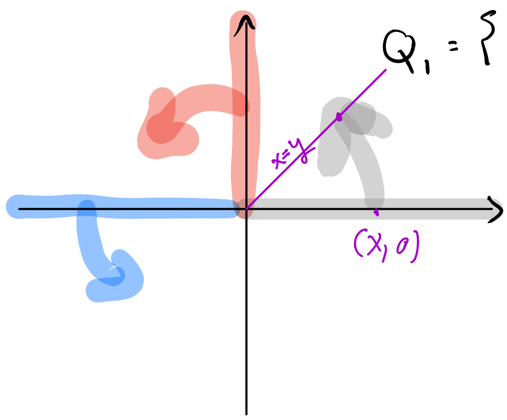
Lemma 1:

$\forall$ finite set $A \subset V$ $\quad |E(A, A^c)| \geq |A|$.

Proof: Let $A_1 = A \cap Q_1$

$Q_1 = \{(x,y) : y \geq 0, x > 0\}$

show $|E(A_1, A^c \cap Q_1)| \geq |A_1|$



- $S(A_1) \subseteq Q_1$
- $T(A_1) \subseteq Q_1$
- $S(A_1) \cap T(A_1) = \phi$

(x,y)

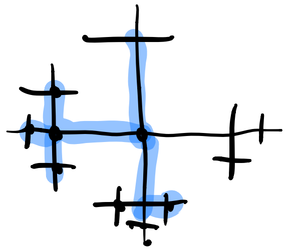
$|S(A_1) \cup T(A_1)| = |S(A_1)| + |T(A_1)| = 2 \cdot |A_1|$

so $E(A_1, A^c \cap Q_1) \geq |A_1|$. ✓

$\square$

wasn't so hard because G is infinite

Another ex: infinite tree

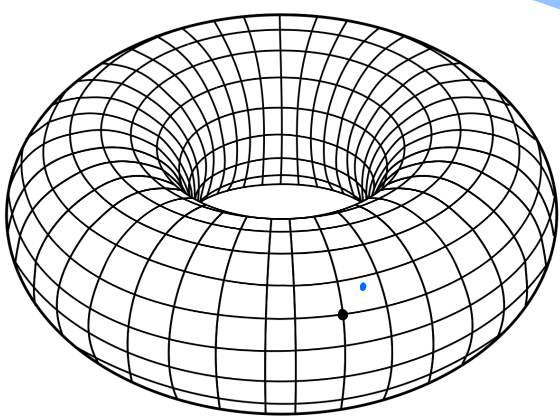


Moving to a infinite family of finite graphs : mod n "quotient"

$$V_n = (\mathbb{Z}/n\mathbb{Z})^2 \qquad E_n = \left\{ (x,y) \right\}$$

(x,y) —
- $S(x,y) = (x+y, y)$, $S^{-1}(x,y) = (x-y, y)$
- $T(x,y)$, $T^{-1}(x,y)$
- $(x \pm 1, y)$ ←
- $(x, y \pm 1)$ ←

"lattice edges" torus



degree of this graph is 8.

How to analyze expansion in $G_n = (V_n, E_n)$

Theorem [M, GG]: G_n is an $(n^2, 8, \lambda)$ expander for some $\lambda > 0$.

Let $\mathbb{T}^2$ be the torus $[0,1)^2$ $\mathbb{R}^2/\mathbb{Z}^2$.

$$L^2(\mathbb{T}^2) = \{ f : \mathbb{T}^2 \to \mathbb{C} \mid \int |f|^2 < \infty \}$$

"$\lambda_2^*(\mathbb{T})$" $:= \min\limits_{\substack{f \in L^2(\mathbb{T}^2) \\ f \neq 0}} \left\{ \dfrac{\|f - f\circ S\|^2 + \|f - f\circ T\|^2}{\|f\|^2} : \int f \cdot 1 = 0 \right\}$

$\langle f, 1 \rangle = 0$

(like a Rayleigh quotient of a Laplacian $Id - M$)

Lemma 2: $\exists\, \varepsilon > 0$ ($\varepsilon = \frac{1}{3}$ works) $\lambda(G_n) \geq \lambda_2(\mathbb{T}) \cdot \varepsilon$

where $\lambda(G_n) := \min\limits_{\substack{f: V_n \to \mathbb{C} \\ f \neq 0 \\ f \perp 1}} \dfrac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum f(u)^2} = f^t L f$

$\sum f(u) = 0 \Leftrightarrow f \perp 1$

Proof: Fix $f: V_n \to \mathbb{C}$ that minimizes the quotient.

Wlog $\sum_{u \in V_n} f(u) = 0$.

$\sum_{u \sim v} (f(u) - f(v))^2 = \sum_{c \in \mathbb{Z}_n^2} (f(c) - f(S(c)))^2 + (f(c) - f(T(c)))^2 + (f(c) - f(c + (0,1)))^2 + (f(c) - f(c + (1,0)))^2$

Define $\tilde{f} \in L^2(\mathbb{T}^2)$ that is similar f. interpolates

$\tilde{f}(x,y) = f(\lfloor nx \rfloor, \lfloor ny \rfloor)$

"floor"

$(\mathbb{Z}/n\mathbb{Z})^2 \leftarrow [0,n)^2 \leftarrow [0,1)^2$

(a) $\int \tilde{f}(z)\, dz = \frac{1}{n^2} \cdot \sum_{u \in V_n} f(u) = 0$

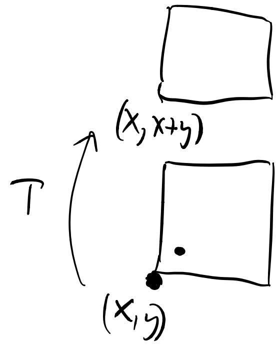
(b) $\|\tilde{f} - \tilde{f}\circ S\|^2 + \|\tilde{f} - \tilde{f}\circ T\|^2 = \int_{[0,1)\times[0,1)} |\tilde{f} - \tilde{f}\circ S|^2 + \ldots$

$= \sum_{(x,y) \in (\mathbb{Z}/n\mathbb{Z})^2} \int_{[x,x+1)\times[y,y+1)} |\tilde{f}(x,y) - \tilde{f}(S(x,y))|^2 + |\tilde{f} - \tilde{f}\circ T|^2 \, dx\,dy$

$\lfloor {}_n(x,y) \rfloor := \lfloor nx \rfloor, \lfloor ny \rfloor$ $\qquad T\lfloor {}_n(x,y) \rfloor = \lfloor {}_n T(x,y) \rfloor$

or

$T\lfloor x,y \rfloor = \lfloor T(x,y) + (0,1) \rfloor$



$(5.1, 11.2)$

$\frac{5.1}{n}$ $\quad \frac{6.1}{n}$

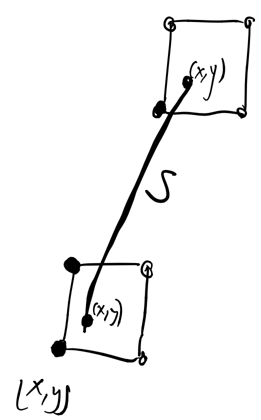
$(5.2, 6.9) \to (5.2, 12.1) \to (5,12)$

$(5,6) \to (5,11)$

$$\sum_{u\sim v}(f(u)-f(v))^2 = \sum_{c\in\mathbb{Z}_n^2} (f(c)-f(S(c)))^2 + (f(c)-f(T(c)))^2 + (f(c)-f(c+(0,1)))^2 + (f(c)-f(c+(1,0)))^2$$

Not hard to show that this is $\geq \frac{n^2}{3} \cdot \|\hat{f} - \hat{f}\circ S\|^2 + \|\hat{f} - \hat{f}\circ T\|^2$.

suppose $|\hat{f}(c) - \hat{f}(S(c))| > L$



Lemma 3: $\lambda_2(\mathbb{T}) > c > 0$ for some constant $c > 0$.

$\mathbb{Z}^2 \setminus (0,0)$ $\quad S, S^{-1} \quad T \quad T^{-1}$

$(x,y) \xrightarrow{S} (x+y, y) \qquad (x,y) \xrightarrow{T} (x, x+y)$

G_n obtained by taking mod n
+ adding $\pm(0,1)$ $\pm(1,0)$
degree 8

$\mathbb{T}$ $\quad [0,1)^2$ $\quad$ we defined a Rayleigh quotient resembling λ_2(Laplacian)

$$\text{“}\lambda_2(\mathbb{T})\text{”} = \min_{\substack{f \in L^2(\mathbb{T}) \\ f \neq 0 \\ \int f = 0}} \frac{\int \left[(f(z) - f \circ S(z))^2 + (f(z) - f \circ T(z))^2 \right] dz}{\int f^2(z) \, dz}$$

$$\underset{\substack{\uparrow \\ \text{the laplacian of } G_n}}{\lambda_2(L(G_n))} \geq \frac{1}{3} \cdot \lambda_2(\mathbb{T})$$

$\underbrace{Id - \frac{1}{8}A_n}$ $\leftarrow$ adj of G_n.

$$\frac{1}{2}\lambda_2(\text{Lap of } G) \leq h_G \leq \sqrt{2\lambda_2}$$

$$\underbrace{\lambda_2(L)}_{\text{edge expansion}}$$

$$\lambda_2(L) = \min_{\substack{f\neq 0\\ f\perp 1}} \frac{f^t L f}{\|f\|^2} = \frac{\sum_{u\sim v}\left(f(u)-f(v)\right)^2}{d\cdot\sum_u f(u)^2}$$

Suppose $f:(\mathbb{Z}_n)^2\to\mathbb{C}$ attains min for G_n

We define $\hat f:\mathbb{T}\to\mathbb{C}$

$$(x,y)\in\mathbb{T} \longrightarrow (\lfloor nx\rfloor, \lfloor ny\rfloor)\in\mathbb{Z}_n^2$$

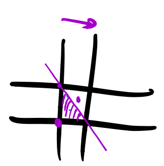
$$\hat f(x,y) := f(\longleftarrow).$$

$$\int_{\mathbb{T}} \hat f(z)^2\, dz = \sum_{(a,b)\in\mathbb{Z}_n^2}\ \int_{[\frac an,\frac{a+1}n)\times[\frac bn,\frac{b+1}n)} \overset{\cancel{\wedge}}{f}(z)^2\, dz = \sum_{(a,b)\in\mathbb{Z}_n^2} f(a,b)^2\cdot\frac1{n^2}$$

$$\int_{\mathbb{T}} (\hat f - \hat f\circ S)^2 + (\hat f - \hat f\circ T)^2 \quad dz$$

if $\quad z$ happens to be $\left(\frac an,\frac bn\right)$ $\qquad \hat f(z) = f(a,b)$

$$\hat f(S(z)) = f(S(a,b))$$



this is true for some z, $\quad (S(z) = S(a,b))$

but sometimes $\quad S(z) = S(a,b)+(1,0)$.

$$\int_{\mathbb{T}}(\hat f(z)-\hat f\circ S(z))^2 = \frac1{n^2}\sum_{a,b}\frac12\left(f(a,b)-f(S(a,b))\right)^2 + \underline{\frac12\left(f(a,b)-f(S(a,b)+(1,0))\right)^2}$$

numerator $\lambda_2(G_n)$: $\sum_{(a,b)\in \mathbb{Z}_n^2} (f(a,b) - f(S(a,b)))^2 + (f(a,b) - f(T(a,b)))^2$
$+ (f(a,b) - f(a,b+1))^2 + (f(a,b) - f(a+1,b))^2$

"Δ ineqs" : $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$

$$\left[f(a,b) - f(S(a,b) + (1,0)) \right]^2$$

$$\leq 2\left[f(a,b) - \underbrace{f(S(a,b))}\right]^2 + 2\left[\underbrace{f(S(a,b))} - f(S(a,b) + (1,0))\right]^2$$

$$\cdots \quad \lambda_2(G_n) \geq \frac{1}{3}\lambda_2(\mathbb{T}).$$

Lemma 3: $\lambda_2(\mathbb{T}) \geq$ " $\lambda_2(\mathbb{Z}^2_{\setminus (0,0)})$ "

where we define $\lambda_2(\mathbb{Z}^2) := \inf_{\substack{f\neq 0 \\ f(0,0)=0}} \frac{\int (f-f\circ S)^2 + (f-f\circ T)^2}{\int f^2}$

$\ell^2(\mathbb{Z}^2) = \{ f: \mathbb{Z}^2 \to \mathbb{C} \mid \sum_{(a,b)\in\mathbb{Z}^2} f(a,b)^2 < \infty \}$

$$\ell^2(\mathbb{Z}^2) \underset{\mathcal{F}}{\overset{?}{\longleftrightarrow}} \ell^2(\mathbb{T})$$

$f: \mathbb{T} \to \mathbb{C}$ $\qquad \hat{f}: \mathbb{Z}^2 \to \mathbb{C}$ $\qquad \hat{f} = \mathcal{F}(f)$

$\int_{\mathbb{T}} f = 0$ $\qquad \hat{f}(0,0) = 0$

- linear
- isometry $\|f\|^2 = \|\hat{f}\|^2$

Recall that every $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{(a,b)\in\mathbb{Z}^2} \hat{f}(a,b)\cdot \chi_{a,b} \qquad (\text{equality in } \ell^2)$$

$$\chi_{a,b}(x,y) := e^{2\pi i(ax+by)}$$

s.t. $\hat{f}(a,b) = \langle f, \chi_{a,b}\rangle = \int_{\mathbb{T}} f(z)\cdot\overline{\chi_{a,b}(z)}\,dz$

$$\chi_{(0,0)} = \overline{1}$$

$$\hat{f}(0,0) = \int f\cdot 1 = \underline{\int f} = 0$$

given

Parseval's identity: $\int_{\mathbb{T}} f^2 = \sum_{\mathbb{Z}^2} \hat{f}(a,b)^2$

$$\lambda_2 > \frac{h^2}{2}$$

$$\inf_{\substack{f\in L^2(\mathbb{T})\\ \int f=0}} \frac{\int (f-f\circ S)^2+(f-f\circ T)^2}{\int f^2}$$

$$\inf_{\substack{\hat{f}\neq 0\\ \hat{f}(0,0)=0\\ \hat{f}\in\ell^2(\mathbb{Z}^2)}} \frac{\sum(\hat{f}-\widehat{f\circ S})^2+(\hat{f}-\widehat{f\circ T})^2}{\sum \hat{f}^2} = \frac{\frac{1}{2}\cdot\sum_{u\in\mathbb{Z}^2}\cdot\sum_{v\sim u}(f(v)-f(u))^2}{\sum f(u)^2}$$

ⓐ $\int f^2 = \sum \hat{f}^2$ ✓

ⓑ $\hat{s}(z) := \hat{f}(z) - \widehat{f\circ S}(z)$; $\quad t(z) := \underline{f(z) - f\circ T(z)}$

$$\int s^2 + t^2$$

Let's calc $\widehat{f\circ S}(a,b) = \langle f\circ S, \chi_{a,b}\rangle$

$$= \int_{x,y} f(S(x,y))\,\overline{\chi_{a,b}(x,y)}\,dz$$

def of S

$$= \int_{\mathbb{T}} f(\underbrace{x+y}_{x'=x+y}, y)\, e^{-2\pi i(ax+by)}\,dx\,dy$$

$$= \int_{\mathbb{T}} f(x', y)\, e^{2\pi i(ax'+(b-a)y)}\,dx'\,dy$$

$$= \langle f, \chi_{a, b-a} \rangle = \underset{\sim}{\hat{f}(a, b-a)}$$

$$\widehat{f \circ \underline{S}}(a,b) = \hat{f}(\underline{T}^{-1}(a,b))$$

similarly, $\widehat{f \circ T}(a,b) = \hat{f}(S^{-1}(a,b))$

$$\hat{s}(z) = \hat{f}(z) - \hat{f}(T^{-1}(z))$$

$$\hat{t}(z) = \hat{f}(z) - \hat{f}(S^{-1}(z))$$

$$\int_{\mathbb{Z}^2} \hat{s}^2 = \int_{\mathbb{Z}^2} \left(\hat{f}(z) - \hat{f}(T(z))\right)^2$$

For every $f \in L^2(\mathbb{T})$ $\quad \int f = 0$

R.Q of f in $\mathbb{T}$ $\underline{\underline{\text{equals}}}$ R.Q. of $\hat{f}$ in $\mathbb{Z}^2$

$$\frac{\int (f - f\circ S)^2 + (f - f\circ T)^2}{\int f^2} = \frac{\int s^2 + \int t^2}{\int f^2} \underset{\substack{\uparrow \\ \text{Parseval} \\ \text{"isometry"}}}{=} \frac{\sum \hat{s}^2 + \hat{t}^2}{\sum \hat{f}^2} = \frac{\sum (\hat{f} - \hat{f}\circ S')^2 + (\hat{f} - \hat{f}\circ T')^2}{\sum \hat{f}^2}$$

$$\rightarrow \quad \lambda_2(\mathbb{T}) \geq \lambda_2(\mathbb{Z}^2) \qquad \square$$

Summarizing: we saw 3 graphs

$$\begin{tikzcd} G_n \arrow[d, bend left] \\ \mathbb{T} \arrow[d, bend left] \\ \mathbb{Z}^2 \end{tikzcd}$$

$\lambda_2(G_n) \geq \frac{1}{3} \cdot \lambda_2(\mathbb{T})$ "discretization/approximation"

Fourier transform: $\lambda_2(\mathbb{T}) \geq \lambda_2(\mathbb{Z}^2)$

expansion here?

$$\lambda_2(\mathbb{Z}^2) \geq \frac{h(\mathbb{Z}^2)^2}{2}$$

Cheeger's inequality

$$\frac{\lambda_2}{2} \leq h \leq \sqrt{2\lambda_2}$$

we saw in finite graphs

we will see.