Zig zag construction of Expander graphs

Two steps: (1) "powering" $G \to G^2$

G^2 is the graph on the same set of vertices, and edges for two-step-walks. More accurately

$$A_{G^2} := A_G \cdot A_G .$$

(also $M_{G^2} = M_G \cdot M_G$, $M_G = \frac{1}{d} \cdot A_G$)

G^2 is d^2-regular

( a self loop counts as adding 1 to degree)

$\vec{1}$

$$M_G \cdot \vec{1} = \vec{1}$$

$$M_{G^2} \vec{1} := M_G \cdot M_G \cdot \vec{1} = \vec{1}$$

Suppose v is an evec of G w eval λ

$$M_G v = \lambda v$$

$$M_{G^2} v = M_G \cdot M_G \cdot v = M_G \cdot \lambda v = \lambda^2 \cdot v$$

(we can remove the d self loops: $A_{G^2} - d \cdot I_n$)

(2) zigzag step: G, H two graphs $G \textcircled{z} H$

$G = (n, m, \alpha)$ - graph

#vertices degree 'two-sided spectral bound: $\max(|\lambda_2|, |\lambda_n|) \le \alpha$

$H = (m, d, \beta)$ - graph

$G ⓩ H = (n \cdot m, d^2, \beta + \max(\alpha, \beta^2))$ - graph

starting point: take H to be $(d^4, d, \frac{1}{4})$ - graph

take $G_1 = H^2$. $(d^4, d^2, \frac{1}{16})$ - graph

$G_2 = (G_1)^2 ⓩ H$ $(d^4 \cdot d^4, d^2, \gamma)$

$\vdots$

$G_{n+1} = (G_n)^2 ⓩ H$

Claim: G_n is a graph on d^{4n} vertices, degree d^2, $\forall n \geq 1$ $\max(|\lambda_2^n|, |\lambda_n^n|) \leq \frac{1}{2}$

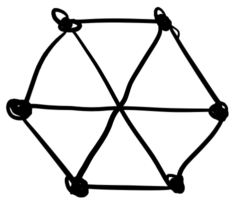
G_n is an $(d^{4n}, d^2, \frac{1}{2})$ - graph.

Before we prove the claim, let's define $G ⓩ H$.

Let G be an (n, m, α) - graph

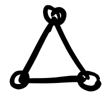
H (m, d, β) - graph.

Define $G ⓩ H$

$G =$ 

$|V_G| = 6$

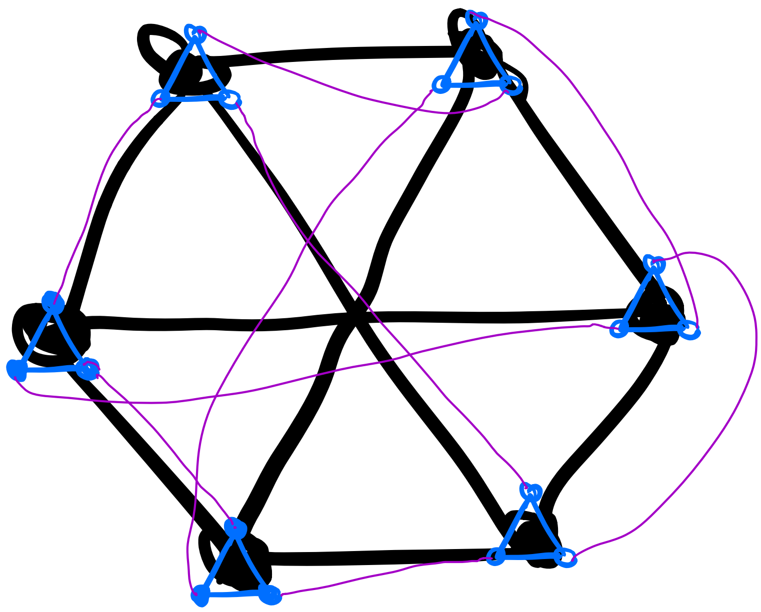
$deg = 3$

$H =$ 

$|V_H| = 3$

$d = 2$

First,

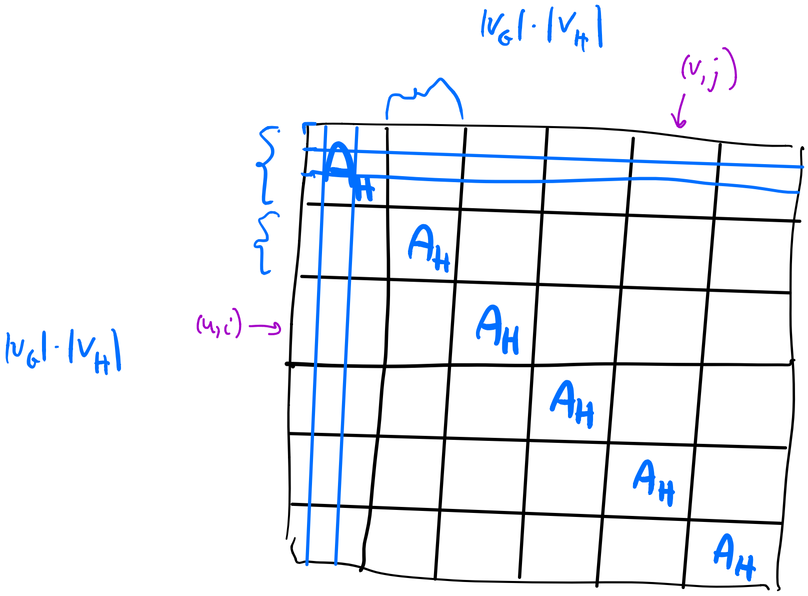


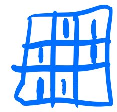
We define $\breve{A}_H$ a matrix for blue steps

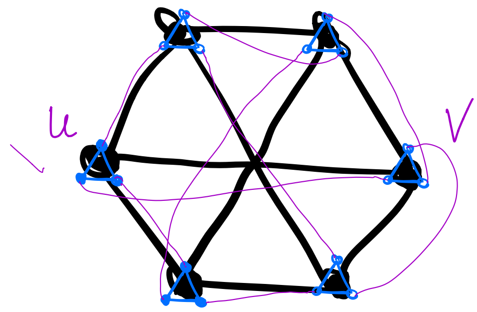
also $\breve{A}_G$ a matrix for purple steps

Our final adj matrix for $G \textcircled{z} H$ will be $\breve{A}_H \breve{A}_G \breve{A}_H$

$\breve{A}_H$



$A_H =$ 



$$\tilde{A}_H = I_{V_G} \otimes \underline{A_H}$$

$$\boxed{\tilde{A}_G\big((u,i),\ (v,j)\big) = 1 \iff u \underset{G}{\sim} v \quad e_i(u) = e_j(v)}$$

$e_i(u)$ is an edge in G

$e_1(u) \dots e_d(u)$ are the incoming edges to u

$$Z := \tilde{A}_H \tilde{A}_G \tilde{A}_H \quad \leftarrow \text{adj matrix of zigzag product}$$

$$M_Z = \frac{1}{d_H^2} Z = \left(\frac{1}{d}\tilde{A}_H\right) \cdot \left(1 \cdot \tilde{A}_G\right) \cdot \left(\frac{1}{d}\tilde{A}_H\right)$$

Lemma: If G is an (n,m,α)-graph and H is an (m,d,β)-graph then $G \text{Ⓩ} H$ is an (nm, d^2, γ)-graph, $\gamma \leq \beta + \max(\alpha, \beta^2)$

Proof: Lets recall:

⊛ $$\lambda = \max_{f \perp \vec{1}} \frac{|\langle Mf, f \rangle|}{\langle f, f \rangle} = \frac{\sum_{i \geq 1} \lambda_i \alpha_i^2}{\sum_{i \geq 1} \alpha_i^2} \quad \Big(\text{where } f = \sum \overset{\text{ev's}}{\alpha_i \cdot v_i}\Big)$$

$(\lambda = \max |\lambda_2|, |\lambda_n|)$.

⊛ (denote $\langle f,g \rangle := \mathbb{E}_v f(v) g(v)$; $\|f\|^2 = \langle f,f \rangle, \dots$)

$$\langle Mf, g \rangle = \mathbb{E}_v (Mf)(v) \cdot g(v)$$

$$= \mathbb{E}_v \left[\mathbb{E}_{u \sim v} f(u)\right] \cdot g(v) = \mathbb{E}_{u \sim v} f(u) g(v)$$

all edges $(u,v) \in E$

Fix some $f : V_Z \to \mathbb{R}$, $f \perp 1$ $\left(\mathbb{E}_{\substack{u \sim V_G \\ i \sim V_H}} f(u,i) = 0\right)$

Define $f^{\parallel}(u,i) = \mathbb{E}_{j \sim V_H} f(u,j)$

Define $f^{\perp}(u,i) = f(u,i) - f^{\parallel}(u,i)$

$$\langle f^{\perp}, f^{\parallel} \rangle = \mathbb{E}_{(u,i)} f^{\perp}(u,i) \cdot f^{\parallel}(u,i) = \mathbb{E}_u \mathbb{E}_i \underbrace{f^{\parallel}(u,i)}_{\substack{\text{doesn't} \\ \text{depend on} \\ i}} \cdot f^{\perp}(u,i) = 0$$

$f = f^{\parallel} + f^{\perp}$

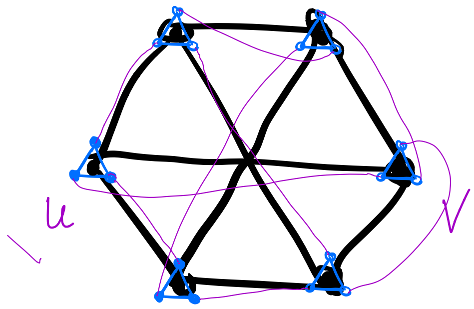
$$|\langle Mf, f \rangle| = |\langle Mf^{\parallel}, f^{\parallel} \rangle + \langle Mf^{\perp}, f^{\perp} \rangle + \langle Mf^{\parallel}, f^{\perp} \rangle + \langle Mf^{\perp}, f^{\parallel} \rangle|$$

=

$$\leq \quad |\langle M f^{\parallel}, f^{\parallel} \rangle|$$

$$+ \quad |\langle M f^{\perp}, f^{\perp} \rangle|$$

$$+ \quad 2|\langle M f^{\parallel}, f^{\perp} \rangle|$$



$$|\langle M f^{\parallel}, f^{\parallel} \rangle| = |\langle \tfrac{1}{d} \tilde{A}_G \underbrace{\tilde{A}_H f^{\parallel}}, \underbrace{\tfrac{1}{d} \tilde{A}_H} f^{\parallel} \rangle|$$

$$= |\langle \underbrace{\tilde{A}_G} f^{\parallel}, f^{\parallel} \rangle| = |\langle M_G \hat{f}, \hat{f} \rangle|$$

where $\hat{f}: V_G \to \mathbb{R}$ $\quad \hat{f}(u) = f^{\parallel}(u,1)$

$\hat{f} \perp 1$

$$\leq \lambda_G \cdot \|\hat{f}\|^2 \leq \lambda_G = \alpha$$

$$\|\hat{f}\|^2 = \|f^{\parallel}\|^2$$

$$\mathbb{E}_{v \in V_G} \hat{f}(v) = \mathbb{E}_{\substack{v \in V_G \\ i \in V_H}} f^{\parallel}(v,i)$$

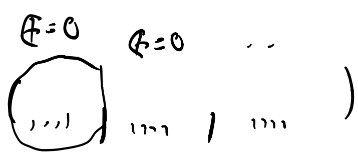
$$= \mathbb{E}_{\substack{v \in V_G \\ i \in V_H}} f(v,i) = 0$$

by choice of f

$$|\langle M f^{\perp}, f^{\perp} \rangle| \leq$$

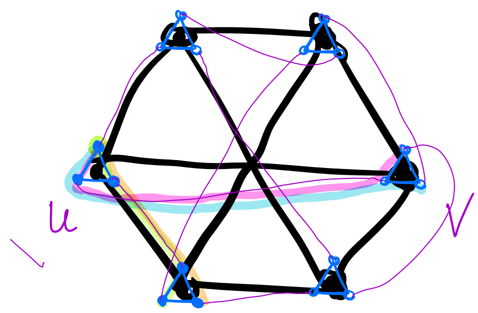
$$f = f^{\parallel} + f^{\perp}$$

$$f^{\parallel} = \hat{f} \otimes \vec{1}_H$$



"replacement product" $G ⓡ H$

"zigzag product" $G ⓩ H$



... cont the proof :

$$|\langle Mf^\perp, f^\perp \rangle| = |\langle \tilde{M}_G \tilde{M}_H f^\perp, \tilde{M}_H f^\perp \rangle| \le$$

normalized adj matrix

$$M = \underbrace{\frac{1}{d_H}\tilde{A}_H}_{\tilde{M}_H} \tilde{A}_G \underbrace{\frac{1}{d_H}\tilde{A}_H}_{\tilde{M}_H}$$

C.S.

$$\le \|\tilde{M}_G \tilde{M}_H f^\perp\| \cdot \|\underbrace{\tilde{M}_H f^\perp}\| \le \|\tilde{M}_H f^\perp\|^2$$

only decreases the norm

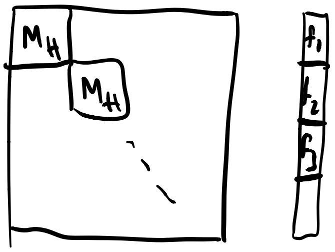
$$\le \lambda_H^2 \cdot \|f^\perp\|^2$$

$$= \beta^2 \cdot \|f^\perp\|^2$$

Recall $\tilde{M}_H = M_H \otimes I_{V_G}$ $\quad \|\tilde{M}_H f^\perp\| \le \lambda_H \cdot \|f^\perp\|$

$$\lambda_{max} = \max \frac{\|Mf\|}{\|f\|} \le 1$$

$\max_{f \neq 0} \frac{\|Mf\|}{\|f\|} =:$ operator norm of M $= \lambda_{max}(M)$

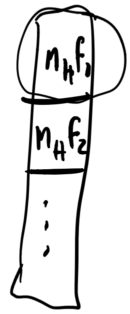


$\mathbb{E} f_1 = 0$

$\mathbb{E} f_2 = 0$

$\vdots$

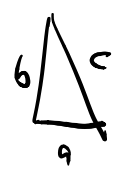
$\tilde{M}_H f^{\perp} =$



$\lambda_H \cdot \|f_1\|$

$$2|\langle M f^{\perp}, f^{\|} \rangle| \le 2 \underbrace{\|M f^{\perp}\|}_{\le \lambda_H \cdot \|f^{\perp}\|} \cdot \|f^{\|}\| \le 2\beta \cdot \underbrace{\|f^{\perp}\| \cdot \|f^{\|}\|}_{\le \|f\|^2}$$

$$\|f\|^2 = \|f^{\|}\|^2 + \|f^{\perp}\|^2$$



$$|\langle Mf, f \rangle| \le \underbrace{\alpha \cdot \|f^{\|}\|^2} + \underbrace{2\beta \|f^{\|}\| \cdot \|f^{\perp}\|}_{\le \text{ AMGM } \beta(\|f^{\|}\|^2 + \|f^{\perp}\|^2) = \beta\|f\|^2} + \underbrace{\beta^2 \|f^{\perp}\|^2}$$

[AMGM : $2xy \le \sqrt{x^2 + y^2}$]

$$= \beta \cdot \|f\|^2 + \alpha(\|f\|^2 - \|f^{\perp}\|^2) + \beta^2 \|f^{\perp}\|^2$$

$$\le \beta \cdot \|f\|^2 + \max(\alpha, \beta^2) \cdot \|f\|^2$$

$\square$

Cayley graphs of (Abelian) groups (& Fourier Transform)

Abelian, finite

Let Γ be a group

$x \in \Gamma$, $y \in \Gamma$ $x+y \in \Gamma$

$x+y = y+x$

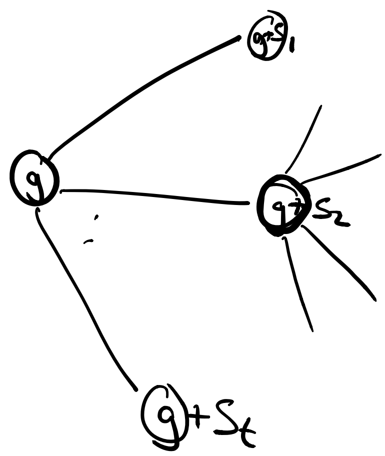
$\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ $x+y \bmod n$ is the operation

$(\mathbb{Z}/2\mathbb{Z})^n = \{0,1\}^n$

Let $S \subset \Gamma$ be a subset closed to inverses $\left(\begin{array}{l} s \in S \\ \Rightarrow s^{-1} \in S \end{array} \right)$

usually not subgroup

The graph $Cay(\Gamma, S)$ has Γ as vertices, and edges are $\{(x, x+s) \mid s \in S\}$.

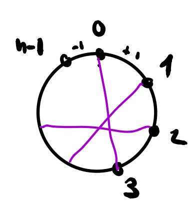


$|S| = t$

Examples: $Cay(\mathbb{Z}_2^n, S = \{e_1 \dots e_n\})$ ← Hamming Cube

$e_1 = (1,0 \dots 0)$

$Cay(\mathbb{Z}_n, S = \{\pm 1\})$ ←



the cycle

+3

<u>Def:</u> (Character). A char. is a homomorphism $\chi : \Gamma \to \mathbb{C} \setminus \{0\}$

i.e. s.t. $\forall\, g,h \in \Gamma$ $\quad \chi(g)\cdot\chi(h) = \chi(g+h)$ $\quad \underset{id_\Gamma}{\chi(0)} = 1$.

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in Γ $\quad \underbrace{g+g+g \cdots + g}_{n\cdot g} = 0$ $\quad$ assuming $|\Gamma| = n$

$$\underline{1} = \chi(0) = \chi(g+\cdots+g) = \chi(g)\cdot\chi(g)\cdots\chi(g) = \big(\underset{\in\,\mathbb{C}}{\chi(g)}\big)^n$$

- $\chi(0) = 1$
- $\chi(a)$ is a root of unity $\forall a \in \Gamma$
- $\alpha = \sum_{a\in\Gamma} \chi(a) = 0$ except if $\chi \equiv \underline{1}$.

Fix $b \neq 0$ s.t. $\chi(b) \neq 1$ $\qquad \underbrace{\sum_{a\in\Gamma} \chi(a+b)}_{\alpha} = \chi(b)\cdot \underbrace{\sum_{a\in\Gamma}\chi(a)}_{\alpha}$

$$\alpha = \chi(b)\cdot\alpha$$

$$\big(\underset{\neq 0}{1-\chi(b)}\big)\cdot\alpha = 0$$

$$\to \quad \alpha = 0$$

- Define inner product $\langle f, g\rangle$ for $f,g : \Gamma \to \mathbb{C}$

$$\langle f, g\rangle = \sum_{a\in\Gamma} f(a)\,\overline{g(a)}$$

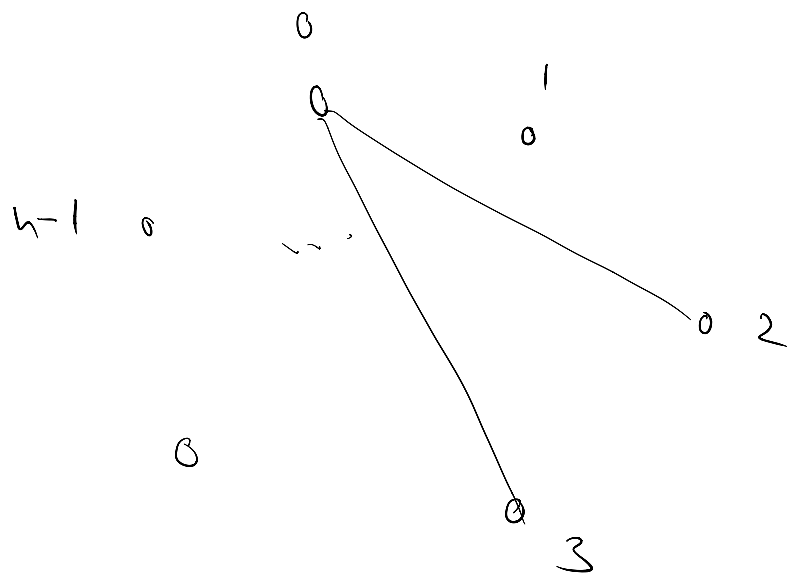
- if χ_1, χ_2 are distinct characters, then

$\langle \chi_1, \chi_2 \rangle = 0$

$\chi_1 \cdot \chi_2$ is also a character

$(\chi_1\chi_2(a) := \chi_1(a)\,\chi_2(a))$

<u>Theorem</u>: Γ abelian group, there are $|\Gamma|$ distinct characters. These are eigenvectors of <u>every</u> Cayley graph on the group Γ (doesn't change when S changes)



<u>Proof</u>: · First look at $\mathbb{Z}_n$

Let $\chi_r : \mathbb{Z}_n \to \mathbb{C}$ be def $\chi_r(a) := e^{2\pi i \frac{r}{n} \cdot a}$

$r = \frac{1}{2}$ $\quad \chi_r(a+b) = e^{2\pi i \frac{r}{n} \cdot (a+b \bmod n)}$

$= e^{2\pi i \frac{r}{n} \cdot a} \cdot e^{2\pi i \frac{r}{n} \cdot b}$

$= \chi_r(a)\,\chi_r(b)$

Notice that $r \neq r'$ $\quad \chi_r(1) \neq \chi_{r'}(1)$

This gives n distinct characters.

this is a character

$$\longrightarrow \quad \langle \chi, \psi \rangle = \sum_{a \in \Gamma} \chi(a)\overline{\psi(a)} = \sum_{a \in \Gamma} \chi\overline{\psi}\,(a) = 0$$

(assuming $\chi \neq \psi$).

$$\left(\chi_r(a)\right)^{|\Gamma|} = \chi_r(\underbrace{a + a + \cdots + a}_{|\Gamma| \text{ times}}) = \chi_r(0) = 1$$

so r must be an integer.

$\chi = \overline{\chi}$

$e^{2\pi i}$

$\chi_r(a) \cdot \chi_r(b) = \chi_r(a+b) \quad \forall a, b$

$a, \; a+a+a$

$a+a$