Def: Let G be a finite abelian group. $\chi : G \to \mathbb{C} \setminus \{0\}$ is a character, if $\forall\, a, b \in G$ $\chi(a)\chi(b) = \chi(ab)$

- $\chi \equiv 1$
- $\forall\, \chi \not\equiv 1$ $\sum_{g \in G} \chi(g) = 0$
- $|\chi(g)| = 1$ $\forall g \in G$ – a root of unity actually.
- $\bar{\chi} = \chi^{-1}$ is also a character
- χ, ψ distinct then $\langle \chi, \psi \rangle = \sum_{g \in G} \chi(g)\overline{\psi(g)} = \sum_g \chi\bar{\psi}(g) = 0$
- if χ, ψ char. then $\chi\psi(g) = \chi(g)\psi(g)$ is a char.

Theorem: Let G be a finite abelian grp. Let S be a set of generators $s \in S \to s^{-1} \in S$.
$Cay(G, S)$ is an undirected graph s.t. the characters form a basis of eigenvectors.

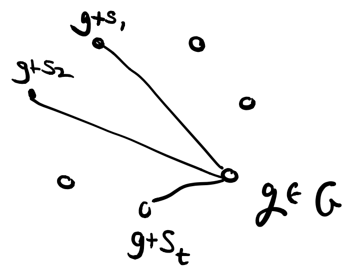
Proof: Vertices of $Cay(G, S)$ are G.
Eigenvectors are functions on the vertex set, and so are the χ's.

First, $G = \mathbb{Z}/n\mathbb{Z}$.
The characters are $\chi_r : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ $\chi_r(u) = e^{\overbrace{2\pi i \frac{r}{n} \cdot u}^{\theta}} = \cos\theta + i \sin\theta$

$r \neq r' \bmod n$ $\chi_r(1) \neq \chi_{r'}(1)$

We found n distinct, pairwise orthogonal vectors.

Let A be the adj matrix of $\mathrm{Cay}(G, S)$. Fix r.



$$A\chi_r(u) = \sum_{v:\, v \sim u} \chi_r(v)$$

$$= \sum_{s \in S} \chi_r(u+s)$$

$$= \chi_r(u) \cdot \sum_{s \in S} \chi_r(s) = \chi_r(u) \cdot \lambda_r$$

where $\underline{\lambda_r} := \sum_{s \in S} \chi_r(s)$.

Remarkable: The eigenvectors <u>don't</u> <u>depend</u> on S.

We've seen that every character <u>is</u> an eigenvector for $\mathrm{Cay}(G, S)$.

Next, we count them.

Suppose $G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \dots$

$$\underbrace{}_{n_1} \qquad \underbrace{}_{n_2} \qquad \dots$$

We claim that a character of $G_1 \times G_2$ is of the form $\chi_1 \cdot \chi_2$, where χ_i: char of G_i.

this gives at most $n_1 \cdot n_2$ characters. (Since G_i has n_i characters by step 1)

No "collisions": if $\left.\begin{array}{l}\chi_r, \chi_{r'} \text{ char for } G_1 \\ \chi_s, \chi_{s'} \text{ char for } G_2\end{array}\right\}$ then $\chi_{r,s} \neq \chi_{r',s'}$

(assuming $r \neq r'$ or $s \neq s'$)

where $\chi_{r,s}(a,b) := \chi_r(a) \cdot \chi_s(b)$

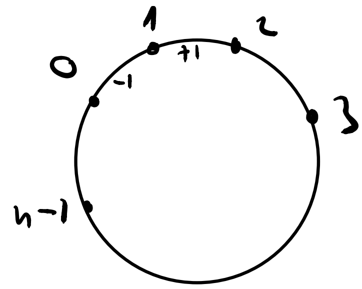
reason if $\chi_r \neq \chi_{r'}$ then $\exists a$ $\chi_r(a) \neq \chi_{r'}(a)$

$$\chi_{r,s}(a,0) = \chi_r(a)\cdot \underbrace{\chi_s(0)}_{1} \neq \chi_{r'}(a)\cdot \underbrace{\chi_{s'}(0)}_{1} = \chi_{r',s'}(a,0).$$

∎

<u>Examples:</u> $\mathbb{Z}/n\mathbb{Z}$ the discrete Fourier Transform

$$\chi_r(u) = e^{2\pi i \frac{r}{n}\cdot u}$$



$\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$

$$f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$$

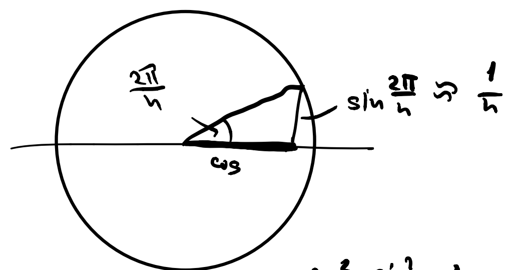
$$f = \sum_{r=0}^{n-1} \hat{f}(r)\cdot \chi_r$$ is the Fourier expansion of f.

Q: What are the eigenvalues?

$$\hat{\lambda}_r = \frac{1}{2}\sum_{s\in S}\chi_r(s) = \frac{1}{2}\chi_r(+1) + \frac{1}{2}\chi_r(-1) = \frac{1}{2}\left(e^{2\pi i \frac{r}{n}\cdot 1} + e^{2\pi i \frac{r}{n}\cdot(-1)}\right)$$

$$= \cos 2\pi \frac{r}{n}.$$

χ_1 has eval $\cos\frac{2\pi}{n} = 1 - \Theta\left(\frac{1}{n^2}\right)$



$\cos^2 + \sin^2 = 1$

$\cos^2\theta = 1 - \frac{1}{n^2}$

$\cos\theta = 1 - \frac{1}{2n^2}$

This gives a <u>tight</u> example for Cheeger's ineq.

$$\frac{1-\tilde{\lambda}_2}{2} \le h \le \sqrt{(1-\tilde{\lambda}_2)\cdot 2}$$

$$\begin{array}{ccccc} \varepsilon & \le & h & \le & \sqrt{\varepsilon} \\ \| & & \| & & \\ \frac{1}{n^2} & \le & \frac{1}{n} & \le & \frac{1}{n} \end{array}$$

$\longleftrightarrow$

tight

The group $G = (\mathbb{Z}/2\mathbb{Z})^n = \{0,1\}^n$ $\quad S = \{e_1, e_2, \dots, e_n\}$

$Cay(G,S)$ = the Hamming cube $\quad x \sim x\oplus e_1,\ x\oplus e_2,\ \vdots,\ x\oplus e_n$

Eigenvectors of this graph are chars of G:

$$\{\underline{\chi_1\cdot\chi_2\cdot\chi_3\cdot \dots \chi_n} : \text{where } \chi_i \text{ char of } \mathbb{Z}/2\mathbb{Z}\}$$

$$\chi_1\cdots\chi_n(\underset{(\mathbb{Z}/2\mathbb{Z})^n}{a_1\dots a_n}) := \chi_1(a_1)\cdot\chi_2(a_2)\cdot \dots \chi_n(a_n)$$

$$\chi_r(u) = e^{2\pi i \frac{r}{2}\cdot u} = (-1)^{u\cdot r}$$

$\chi_0 \equiv 1$

$\chi_1 \equiv (\underset{u=0}{1}, \underset{u=1}{-1})$

$$\chi_{r_1 r_2 \dots r_n}(a_1\dots a_n) = (-1)^{r_1\cdot a_1}\cdot(-1)^{r_2\cdot a_2}\cdot(-1)^{r_3\cdot a_3}\dots$$

$r_i \in \{0,1\}$

$$= (-1)^{\underbrace{\sum r_i a_i}_{r \cdot a} \bmod 2}$$

$$\chi_{11111}(a_1 \ldots a_n) = (-1)^{a_1 + a_2 + \ldots + a_n \bmod 2}$$

$$\chi_{110\ldots0}(a_1 \ldots a_n) = (-1)^{1 \cdot a_1 + 1 \cdot a_2 + \cancel{0 \cdot a_3 + 0 \cdot a_4 \ldots}} = (-1)^{a_1 + a_2 \bmod 2}$$

Every function $f: \{0,1\}^n \to \mathbb{C}$ can be written as

$$f = \sum_{\substack{r \\ \text{subset of } [n]}} \hat{f}(r) \cdot \chi_r$$

Eigenvalues:

$$\tilde{\lambda}_r = \frac{1}{n} \sum_{s \in S} \chi_r(s) = \frac{1}{n} \sum_{i=1}^{n} (-1)^{r \bullet e_i} = \frac{1}{n} \sum_{i=1}^{n} (-1)^{r_i} = \frac{1}{n}\left(-|r| + 1 \cdot (n - |r|)\right)$$

$$= 1 - \frac{2|r|}{n}.$$

For $\vec{r} = (1, 0 \ldots 0)$ (or any $|\vec{r}| = 1$) $\lambda_{\vec{r}} = 1 - \frac{2}{n}$

$$\chi_{\vec{r}}(a_1 \ldots a_n) = (-1)^{\sum r_i a_i \bmod 2} = (-1)^{a_1} \leftarrow \text{"dictator"}$$

$\vec{r} = (1, 0 \ldots 0)$

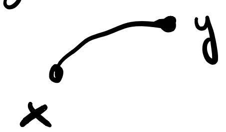
$$\frac{1 - \tilde{\lambda}_2}{2} \leq h \leq \sqrt{(1 - \tilde{\lambda}_2) \cdot 2}$$

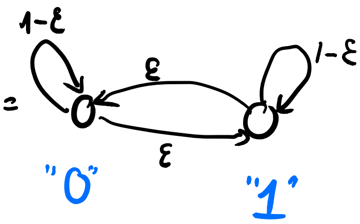
$\frac{1}{n}$ $\longleftrightarrow$ tight $\frac{1}{n}$

H_n – Hamming cube $\mathrm{Cay}\left((\mathbb{Z}/2\mathbb{Z})^n, \{e_1, \dots, e_n\}\right)$

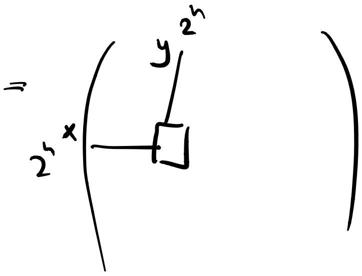
$N^n_{\varepsilon_f}$ – ε-noisy hypercube $\left(V = \{0,1\}^n, \quad E = \dots\right)$

edges corr to "noise process"



$G =$ 

$M_G = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}$

$M_G^{\otimes n} =$ 

$(1-\varepsilon)^{n-d} \cdot \varepsilon^d$ where $d = \mathrm{dist}(x,y)$

$\chi_{\vec{r}}(a_1, \dots, a_n) = (-1)^{\sum r_i a_i}$

$\chi_r = \prod \big(\underbrace{(-1)^{r_i}}_{X_i}\big)^{a_i}$ ← monomial

$f =$ poly in $X_1 \dots X_n$ $\quad X_i$ ± 1 variable

$$f(x_1 \dots \underset{\in\{\pm 1\}}{x_n}) = \sum_r \hat{f} \underset{|r|}{\prod_{j \in r} x_j}$$

$$f(x_1 \dots x_n) = x_1$$

$$f(x_1 \dots x_n) = \frac{1}{n} \sum x_i \sim$$