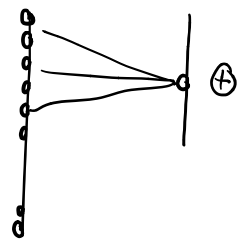
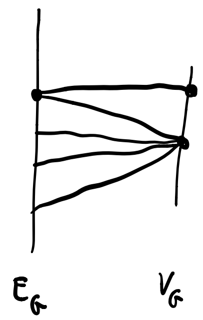
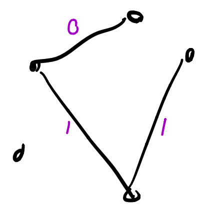
Tanner / Sipser-Spielman Codes



Let G be a d-reg graph $G=(V_G, E_G)$



bip. view of G



G

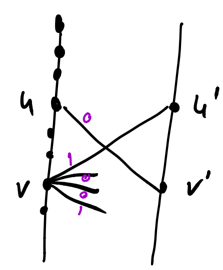
Let $C_0 \subseteq \{0,1\}^d$ with distance $\delta_0 d$. Linear code.

Tanner code $G(C_0) = \{ w : E \to \{0,1\} \mid \forall v \;\; w|_{E(v)} \in C_0 \}$

$C_0 \subseteq \{0,1\}^{E(v)}$

Def: The double cover of G is a bip. graph G_2

$(V_G \cup V_G', \; E = \{(u,v'), (u',v) \mid u \sim_G v\})$



#edges $n \cdot d$
#verts $2 \cdot n$

Rate: Suppose $\dim C_0 = \underbrace{r}_{\substack{\text{relative}\\ \text{rate}}} \cdot d$

$\#$ lin. constraints per vertex: $d - rd$.

constraints in $G_2(C_0)$ $\quad 2\cdot n\cdot(d-rd)$

$\dim G_2(C_0) \geq nd - 2nd + 2nrd = nd(2r-1)$

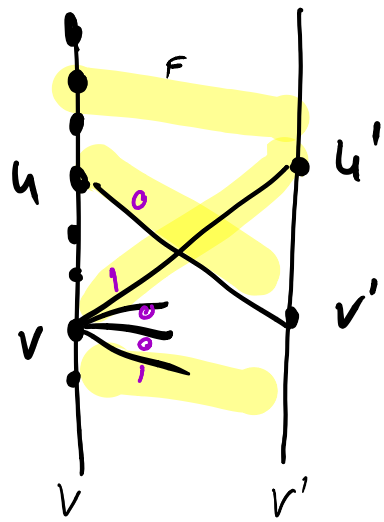
if $r > \frac{1}{2}$ $\quad G_2(C_0)$ has $\Rightarrow$ rate > 0.

<u>Distance</u>: Let $w \neq 0$ codeword. Let $F = \{(a,b) \mid w[(a,b)] = 1\}$

$w : E \to \{0,1\}$

$S = \{u \in V \mid u \text{ touches an edge in } F\}$

$T = \{u' \in V' \mid - - \text{———}\}$



<u>EML</u>: $S \subseteq V$, $T \subseteq V'$ $\quad \left| E(S,T) - |S|\cdot|T|\cdot\frac{d}{n} \right| \leq d\cdot\lambda\sqrt{|S|\cdot|T|}$

max$(|\lambda_2|, |\lambda_n|)$ normalized
of G

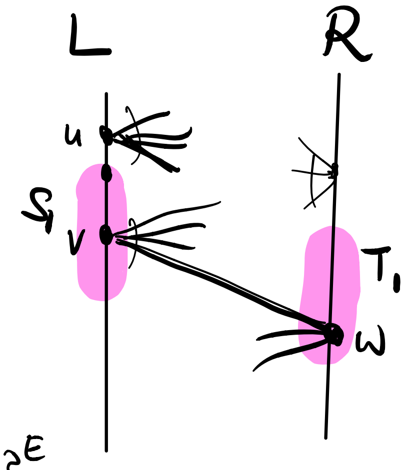
$$\delta_0 d\cdot\sqrt{|S||T|} \leq \max(\delta_0 d\cdot|T|, \delta_0\cdot d\cdot|S|) \leq |F| \leq |E(S,T)| \leq |S|\cdot|T|\cdot\frac{d}{n} + d\lambda\sqrt{|S||T|}$$

$$d\cdot(\delta_0 - \lambda) \leq \sqrt{|S.T|}\cdot\frac{d}{n} \leq \frac{d}{n}\cdot\frac{|F|}{\delta_0 d}$$

$$nd\,\delta_0\,(\delta_0 - \lambda) \leq |F|$$

$\square$

Lemma 17 *Assume that* $\lambda/\cancel{d} < \delta_0/3$. *When the number of errors is at most* $(1-\epsilon)\frac{\delta_0}{2}(\frac{\delta_0}{2} - \frac{\lambda}{d})nd$, *the above algorithm converges to the correct codeword when run for* $A(\epsilon)\log n$ *iterations.*



ALG

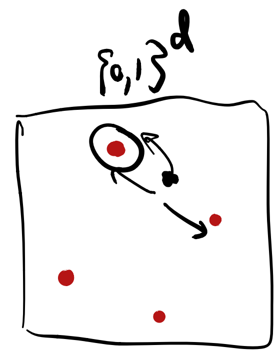
All left decode (local-correct)
All right decode
repeat.

input: $y \in \{0,1\}^E$
"error" means bit that differs from $w_0 \in C$ the nearest codeword.

$$F = \{ e \in E \mid \underset{\text{closest codeword}}{w(e)} \neq \underset{\text{received}}{y(e)} \}$$

$$S_1 = \{ u \in L \mid u \text{ sees an error after the first left-decoding} \}$$

every $u \in S_1$ sees $\geq \frac{\delta_0}{2} d$ errors



$$|S_1| \cdot \frac{\delta_0 d}{2} \leq |F| \leq (1-\varepsilon)\frac{\delta_0}{2}\left(\frac{\delta_0}{2} - \frac{\lambda}{d}\right)nd$$

$$\rightarrow |S_1| \leq (1-\varepsilon)\left(\frac{\delta_0}{2} - \frac{\lambda}{d}\right) \cdot n$$

$$T_1 = \{ v \in R \mid v \text{ sees an error after the first right decoding} \}$$

Claim: $|T_1| \leq \dfrac{|S_1|}{1+\varepsilon}$

$$|T_1| \cdot \frac{\delta_0 d}{2} \underset{\text{errors } \leq}{\leq} E(S_1, T_1)$$

$$E(S_1, T_1) \leq \frac{d}{n}|S_1| \cdot |T_1| + \lambda d \sqrt{|S_1| \cdot |T_1|}$$

$$\frac{\delta_0 d}{2} \cdot |T_1| \leq |T_1| \cdot \left(\frac{d\delta_0}{2} - d\lambda\right)(1-\varepsilon) + \lambda d \, \frac{|S_1| + |T_1|}{2}$$

$$|T_1| \leq \frac{\lambda d}{\varepsilon \delta_0 d + (1-2\varepsilon)\lambda d} \cdot |S_1| \leq \frac{|S_1|}{1+\varepsilon}$$

$\lambda \leq \frac{\delta_0}{3}$.

Conclusion: after 1 step, #error vertices shrinks by $\frac{1}{1+\varepsilon}$

~ after $\log_{1+\varepsilon} n$ steps we converge.

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rate, distance. $\rightarrow r_0 > \frac{1}{2} + \varepsilon$ $\quad 2r - 1 \rightarrow 2\varepsilon$

$\rightarrow \delta_0$ $\quad \underline{\delta_0(\delta_0 - \lambda)}$

$h(\delta_0) + r > 1 - \varepsilon$