Codes with optimal parameters
(Ta-Shma's ε-biased sets)

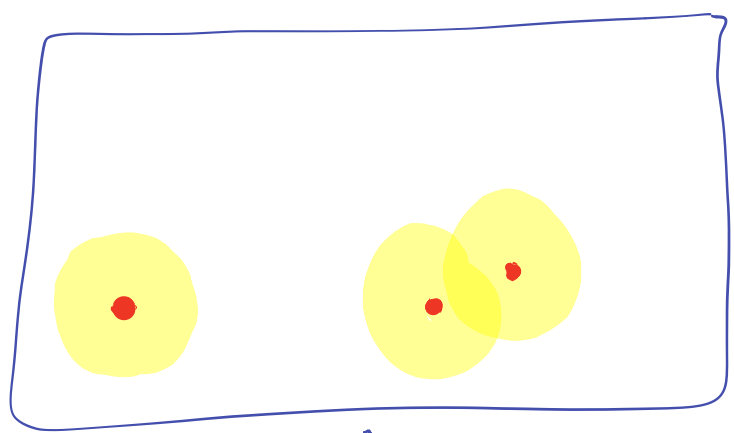
$C \subseteq \{0,1\}^n$

$$C = G[C_0]$$

expander   base code

$\delta_0(\delta_0+\lambda)$ δ_0

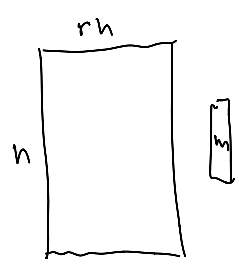
$2r_0 - 1$ r_0



$\{0,1\}^n$

$$2^{rn} \cdot 2^{nh(\delta)} < 2^n$$

$\rightarrow$ GV : $\exists$ code s.t. $r + h(\delta) \geq 1 - o(1)$



$\delta \approx \frac{1}{2}$ $\delta = \frac{1}{2} - \varepsilon$ GV bound gives $r \approx \varepsilon^2$

$2^{n \cdot \varepsilon^2}$ strings $\forall$ pair dist $\geq \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) n$

$\leq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) n$

distance amplification:

suppose $w \in \{-1,1\}^n$ bias $(w) = \varepsilon_0$ $\frac{1}{2} + \frac{\varepsilon_0}{2}$

$\| \quad \frac{1}{n}\left|\mathbb{E}_i w_i\right|$

$f = w \otimes w \in \{\pm 1\}^{n^{\overset{t}{2}}}$ $f(i,j) = w_i \cdot w_j$

$\underbrace{w \otimes \cdots \otimes w}_{t \text{ times}}$

bias $(f) = \frac{1}{n^2}\left|\mathbb{E}_{ij} f(i,j)\right| = \frac{1}{n^2}\left|\mathbb{E}_i w_i \cdot \mathbb{E}_j w_j\right|$

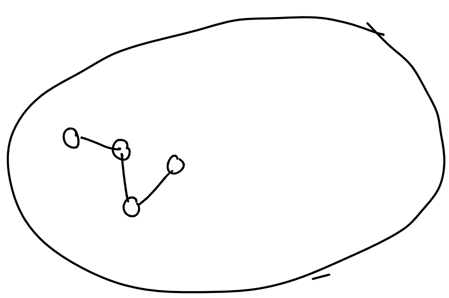
$= (\text{bias}(w))^2 = \varepsilon_0^2$ ε_0^t t

subsampling: choose $\sim \left\{ \frac{n}{\varepsilon^2} \text{ random tuples } i_1, \dots, i_t \right\} = I$

$\left(w^{\otimes t} (i_1, \dots, i_t) \right)_{(i_1, \dots, i_t) \in I}$

derandomized version

$t = 2$: choose i,j edges on an n-vertex expander



$G = ([n], E)$ $C_1 \subseteq \{0,1\}^n$ ε_0

d-regular

λ $C_2 \subseteq \{0,1\}^E$

$$C_2 = \left\{ w : E \to \{0,1\} \;:\; \exists f \in C_1 \text{ s.t. } w(u,v) = f(u) \oplus f(v) \;\; \forall\, uv \in E \right\}$$

(if G were clique graph, $C_2 = \{ \underline{f \otimes f} : f \in C_1 \}$

$$C_t = \left\{ w : \underbrace{\text{Paths}(t)}_{n \cdot d^{t-1}} \to \{0,1\} \;\middle|\; \exists f \in C \text{ s.t. } \forall \text{ path in } G \; v_1 \dots v_t \;\; w(v_1 \dots v_t) = \underbrace{f(v_1)}_{\{0,1\}} + f(v_2) \dots + f(v_t) \right\} \text{ mod } 2.$$

Theorem: Fix $f \in \{0,1\}^n$ $\text{bias}(f) = \varepsilon$

$$\text{bias}(\underbrace{\text{PATH}(f)}) \leq (\underline{\varepsilon} + 2\lambda)^{\lfloor t/2 \rfloor}$$

where $\text{Enc}(f)$ maps to every t-walk on G a bit

$$\text{PATH}(f)\,[v_0 \dots v_t] := f(v_0) + \dots + f(v_t) \text{ mod } 2.$$

$\#$ codewords $= |C_t| = |C|$

$\#$ blocklength $= n \cdot d^t$

$$\frac{r \cdot n}{n \cdot d^t} \qquad r(C_t) = \frac{r}{\underline{d^t}}$$

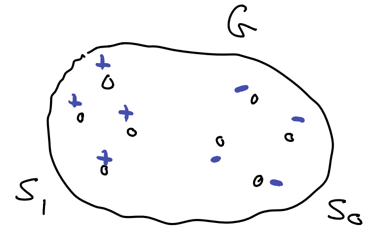
if $t = c \cdot \log \frac{1}{\varepsilon}$ get $\text{bias}(C_t) = \varepsilon$

rate $\frac{1}{\varepsilon^{o(1)}}$ $\qquad \frac{1}{\varepsilon^4}$

Proof: $f \in C$ $\quad \text{bias}(\underbrace{\text{PATH}(f)}_{\text{encoding of } f}) \leq (\text{bias}(f) + 2\lambda)^{t/2}$

$$S_0 = \{ v \in V(G) \mid f(v) = 0 \}$$

$$S_1 = \{ v \in V(G) \mid f(v) = 1 \}$$



Let $\Pi_0 = \begin{array}{c} S_0\{ \end{array}\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix}$ $\quad \Pi_1 = \begin{pmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}$

$$\underbrace{\vec{1}^t}\, \Pi_1 G \Pi_0 \vec{1} = \text{\# edges from } S_0 \text{ to } S_1$$

$$\Pi_0 G \Pi_0$$

$$S_0 \to S_1 \to S_0$$

$$\vec{1}^t\, \Pi_0 G \Pi_1 G \Pi_0 \,\vec{1} = \text{\# 2-walks } S_0 \text{ to } S_1 \text{ to } S_0.$$

$$\circledast \quad \underbrace{\left| \sum_{b_0, b_1, b_2 = 0,1} (-1)^{\Sigma b_i} 1^t\, \Pi_{b_0} G \Pi_{b_1} G \Pi_{b_2} 1 \right|} = \text{bias of length 2 walks.}$$

$$\Pi = \Pi_0 - \Pi_1 \qquad \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & -1 \end{pmatrix}$$

note that $\Pi \vec{1} = f$ in $\pm$ notation

$$\vec{1}^t\, \Pi G \Pi \,\vec{1}$$

$$\circledast = 1^t\, \Pi \underbrace{G \Pi} \underbrace{G \Pi}\, 1$$

more general t: $\displaystyle\sum_{b_0 \ldots b_t = 0,1} (-1)^{\Sigma b_i}\, \Pi_{b_0} G \Pi \ldots \Pi = 1^t (\Pi G)^t \Pi 1$

<u>Claim:</u> $\| \Pi G \Pi G \| \leq (\text{bias}(f) + 2\lambda)$

Let v be a norm 1 vector. We represent $v = v^{\|} + v^{\perp}$. Notice that $Gv^{\|} = v^{\|} = \left\|v^{\|}\right\|\mathbf{1}$. Therefore,

$$
\begin{aligned}
\|\Pi G \Pi G v\| &\leq \left\|\Pi G \Pi G v^{\|}\right\| + \left\|\Pi G \Pi G v^{\perp}\right\| \\
&\leq \left\|v^{\|}\right\| \|\Pi G \Pi \mathbf{1}\| + \|\Pi G \Pi\| \left\|G v^{\perp}\right\| \\
&\leq \|\Pi G \Pi \mathbf{1}\| + \left\|G v^{\perp}\right\| \\
&\leq \left\|\Pi G (\Pi \mathbf{1})^{\|}\right\| + \left\|\Pi G (\Pi \mathbf{1})^{\perp}\right\| + \left\|G v^{\perp}\right\| \\
&\leq \left\|(\Pi \mathbf{1})^{\|}\right\| + 2\lambda. \quad \leq \text{bias}(f) + 2\lambda
\end{aligned}
$$

$z = \Pi\mathbf{1} = (1111\ -1-1-1-1)$

$z^{\|}$ $z^{\perp}$

To finish the proof note that $\underbrace{\left\|(\Pi \mathbf{1})^{\|}\right\|}_{z^{\|}} = |\langle \overbrace{\Pi \mathbf{1}}^{z}, \mathbf{1} \rangle| = |\frac{|S_0| - |S_1|}{n}|$.

$= \text{bias}(f)$