Examples: 

**3SAT**

\[ x_1, \ldots, x_n \]

\((\neg x_1 \lor x_2 \lor x_3) (\neg x_2 \lor x_3 \lor \neg x_8) \ldots m \text{ clauses}\)

**K-LIN**

\[ x_1, \ldots, x_n \quad x_i \in \mathbb{F}_2 \]

\[ \begin{align*}
Ax &= b \\
\sum_{i=1}^{k} x_i &= 0^v \\
\sum_{i=1}^{k} x_i &= 1^v \end{align*} \]

deciding if 100% is SAT is \( \in \mathbb{P} \)

deciding if \( \geq 99\% \) is SAT is \( \mathbb{NP} \)-hard.

for \( k = 2 \) & all RHS = 1

\[ \begin{align*}
x_1 + x_2 &= 1 \\
x_2 + x_3 &= 1 \\
& \vdots \end{align*} \]

is there a cut cutting \( \geq 99\% \) of edges? \( \mathbb{NP} \)-hard

**t-COLORING**

given a graph, is there a \( t \)-coloring \( x: \text{V} \rightarrow [t] \)

s.t. all edges see different colors?

\((\mathbb{NP} \text{- hard } t \geq 2)\)

\[ x_1, \ldots, x_n \quad [t] \]

\[ \begin{cases}
\forall i, j \in E : x_i \neq x_j \\
\forall i \neq j \end{cases} \]
CSP as a maximization problem:
find an assign. to vars, sat max num of constraints.

\[ \Phi \] family of constraints

\[ x_1 + \ldots + x_k = 0 \leftarrow \]
\[ x_1 + \ldots + x_k = 1 \leftarrow \]

A CSP instance wrt \( \phi \):
vars \( x_1, \ldots, x_n \)
k-tuples \( (x_i, \ldots, x_k) \)

\[ \rightarrow \max - 3SAT \]
\[ \rightarrow \max - \text{KLIN} \]
\[ \rightarrow \max - \text{cut} \ (\max 2\text{LIN}) \]

PCP then \( \rightarrow \) all of these are hard to qpx \((1-\delta)\alpha \)

3SAT: even if promised \( \alpha = 1 \) (\( \exists \) assign sat 100%)

Thm (RP-hard)
no alg can sat \( \geq \frac{7}{8} + \epsilon \) in a 3SAT instance

no \( \geq \frac{1}{2} + \epsilon \) in a KLIN instance \((k \geq 2)\).

"98%"

max-cut: SDP algorithm \((GW)\) satisfies \(1 - \sqrt{\delta}\) of edges
when we know \( \exists \) sol w vol \( \geq 1 - \delta \).
input \( w \in \{0,1\}^n \cdot d^{k-1} \)
goal: find \( z \in \{0,1\}^n \) s.t. \( z(v_1) + \ldots + z(v_k) = w(v_1, \ldots, v_k) \mod 2. \)

**Thm (Granith Jeronimo Quintana, Srivastava, Tulsiani 2020)**

if \( k \)-lin instance is "splittable" then we can approximate very well.

\[
\begin{align*}
V_1 & \quad V_{k-2} & \quad V_k \\
\frac{k}{2} & \quad \frac{k}{2} & \quad \frac{k}{2}
\end{align*}
\]

\[
\begin{align*}
V & \quad V' \\
\frac{k}{2} & \quad \frac{k}{2}
\end{align*}
\]

\[
\begin{align*}
\frac{k}{2} & \quad \frac{k}{2}
\end{align*}
\]

→ How to solve 2-CSPs on expanders?

**Unique games CSP:**
- constraints are 1-to-1 permutations
- alphabet \([r]\)

BTW \( 2-LIN \) is a "unique games" CSP \( x_i + x_j = 1 \)

Unique games conj (Khot '02)

\( (2-LIN) \)
it is \( \text{NP-hard} \) to decide if a given UG instance

\[ \text{val} \geq 1 - \delta \quad \text{"99%"} \]
\[ \text{val} \leq \delta \]

where are hard instances of CSPs? of UG CSPs?

Random instances?

\[ \rightarrow \text{Feige R3SAT conj - hard to decide between} \]

\[ \text{random instances} \]
\[ \text{planted instances} \]

Thm: UG are easy on expander graphs \([\text{Arora-Khot-Kolla-Steurer-Tulsini-Vishnoi}]_2008\)

(C in particular, on random instances).

Let \( G \) be the constraint graph. \( \lambda_2 = \lambda_2(G) \) (normalized)

\[ \exists \text{alg s.t. if } \text{val} \geq 1 - \varepsilon \]

\[ \text{alg finds an assign w val } 1 - \frac{\varepsilon}{(1-\lambda_2)} \cdot \log \frac{1}{1-\lambda_2} \]

\[ G' \]

\( G \) label-extended graph

\( (u_i, j) \quad (v_i, j) \)

\( G' \) is an \( r \)-lift of \( G \).
Claim: an assignment \( w, val \geq 1 - \varepsilon \) gives a cut in \( G' \) with \( \varepsilon \) frac of edges of \( S \) leaving \( S \).

Proof:

Example: \( G \) is a \( k \)-cycle.

\[ G' \] is a \( 3k \)-cycle.

Proof: Suppose \( G \) UG instance, let \( \alpha : V \rightarrow [r] \) assign

\[ \text{Prob}_{\alpha} \left[ \left( a(w), a(v) \right) \in \Pi_{uv}^{*} \right] \geq 1 - \varepsilon \]

Recall \( V(G') = V \times [r] \)

Let \( S = \{ (v, a(v)) : v \in V_G \} \).

Let \( G^{*} \) be a perturbed UG instance s.t. \( \alpha \) has \( val = 1 \).

Clearly \( S \) is disconnected in \( (G^{*})' \) (the label extended graph of \( G^{*} \))

The only edges leaving \( S \) in \( G' \) are above edges \( E^{*} = \) edges falsified by \( \alpha \).
Converse: Given a UG instance $G$.
Let $G'$ be the L.E. graph. A sparse cut in $G'(S, \overline{S})$
\text{ s.t. } S \text{ intersects each fiber once } \rightarrow \text{ no good assignment to } G.$

Special case of UG. $2\text{-LIN}(r)$.

Claim: Let $a^i : V \rightarrow [r]$ \text{ s.t. } a^i(a+1) \equiv a \pmod{r}$
\text{val}(a) = \text{val}(a')$

$S_a = \{ (v, a(v)) : v \in V \}$

$S_a, S_{a+1}, S_{a+2}, \ldots, S_{a+r-1}$

ALG: Given $G$ expending $2\text{-LIN}(r)$ instance.
- Construct $G'$.
- Find $r$ top eigenvectors for $A_{G'}$.

Analysis:  
Define $z_0, \ldots, z_{r-1}$ indicator for $S_{q+i \mod r}$

Claim: $z_i$ are eigenvectors for $G^\wedge$, $\omega$ eval 1.

of course, lin. ind. (and orthg.)

also, their Rayleigh quot is $\geq 1-2\varepsilon$

$\Theta^*$ has at least $r$ eigenvalues above $1-2\varepsilon$.

Let $f$ be an eigenvector of $G^\wedge$, $f \perp z_1, \ldots, z_r$.

$f^* A_{G^\wedge} f = \lambda_2 (A)$

can write $f = f_1 + f_2 + \ldots + f_r$

$\uparrow$

zero outside $S_{q+i}$

$\downarrow$

zero outside $S_{q+i}$

RQ of $f$ is normalized sum of RQ's of $f_i$, $\leq \lambda_2 (A)$.

\[
\sum_{i=1}^{r} \lambda_2 (z_i)
\]

(How to solve $K$-LIN on instances "splittable to expanders")

Regularity Lemmas

$\forall G = (V,E) \forall \varepsilon \exists \text{ \text{ partition}}$

$\pm \varepsilon n^2$ edges

(Szemeredi reg. ) $\text{tower}(\frac{1}{\varepsilon^2})$
useful for CSPs on dense instances.

Reg lemma for "splittable tuples"