Constraint Satisfaction Problems (CSPs)

Examples: 3SAT $x_1 \; \ldots \; x_n$

$(\neg x_1 \vee x_2 \vee x_3)(\neg x_2 \vee x_7 \vee \neg x_8) \ldots$

m clauses

k-LIN $x_1 \; \ldots \; x_n$ $x_i \in \mathbb{F}_2$ field

$$\left.\begin{aligned} x_1 + x_2 + \ldots + x_k &= 0 \\ x_1 + x_3 + \ldots + x_{17} &= 1 \end{aligned}\right\} \text{linear eqs. each } \omega \; k \text{ vars}$$

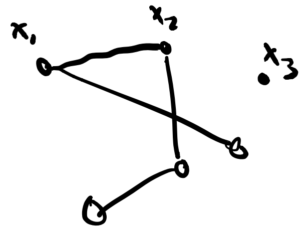
$Ax = b$

deciding if 100% is SAT is $\in P$

deciding if $\geq 99\%$ is satis. is NP-hard.

for $k = 2$ & all RHS $= 1$

$$\left\{\begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_7 &= 1 \\ &\vdots \end{aligned}\right. \qquad \mathbb{F}_2$$



is there a cut cutting $\geq 99\%$ of edges? NP-hard

t-COLORING : Given a graph, is there a coloring $x : V_G \to [t]$ s.t. all edges see diff. colors?

(NP-hard $t > 2$).

$x_1 \; \ldots \; x_n$ $[t]$

$\{ x_i \neq x_j \quad ij \in E$

CSP as an maximization problem :

find an assign. to vars, sat max num of constraints.

→ Φ · family of constraints $x_1+\dots+x_k=0$ ←
$x_1+\dots+x_k=1$ ←

A CSP instance wrt ϕ : vars $x_1 \; - - - \; x_n$

k-tuples $(x_{i_1}, .. x_{i_k}), - - - -$

→ max - 3SAT

→ max - KLIN

→ max - cut (max 2LIN)

PCP thm → all of these are hard to apx $(1-\varepsilon)\cdot\alpha$

promise ∃ solution w value α

3SAT : even if promised $\alpha=1$ (∃ assign sat 100%)

Thm (PCP thm. Hastad) no alg can sat $\geq \frac{7}{8}+\varepsilon$ in a 3SAT instance

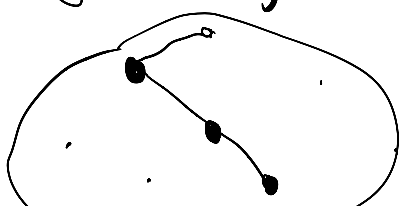
no - - - - - - $\geq \frac{1}{2}+\varepsilon$ - - KLIN instance $(k>2)$.

"98%"

max-cut : SDP algorithm (GW) satisfies $1-\sqrt{\varepsilon}$ of edges

when we know ∃ sol w val $\geq \underbrace{1-\varepsilon}_{99\%}$.

G d-regular

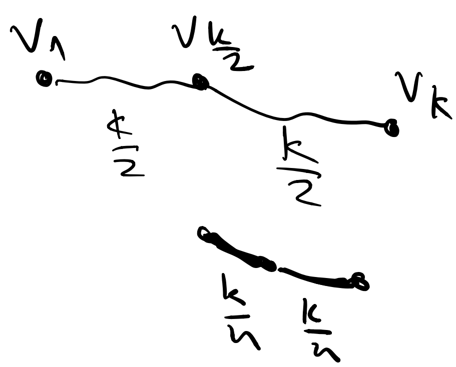


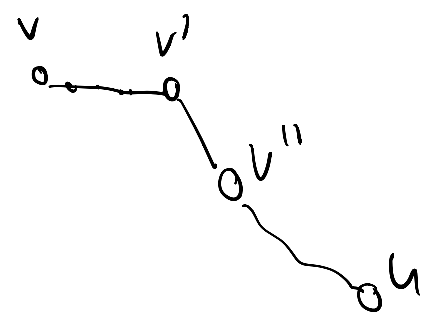
∀ walk $v_1 .. v_k$ $w[v_1..v_k] \in \mathbb{F}_2$

input $w \in \{0,1\}^{n\cdot d^{k-1}}$

goal: find $z \in \{0,1\}^n$ s.t. $z(v_1) + \dots + z(v_k) = w(v_1 \dots v_k) \bmod 2$.

<u>Thm</u> (Granha Jeronimo Quintana Srivastava Tulsiani (2020))
if k-lin instance $\overset{\lambda}{\text{is}}$ "splittable" then we can approximate very well! r

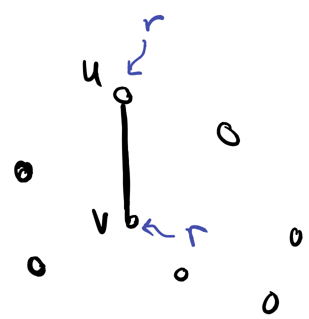




→ How to solve $\overset{\text{some}}{}$ 2-CSPs on expanders ?

<u>Unique games CSP</u> :

constraints are 1-1 permutations



alphabet $[r]$

BTW ~~so~~ 2-LIN is a "unique games" CSP $\quad x_i + x_j = 1$ $\quad \mathbb{F}_p$

Unique games conj (Khot '02)

(2-LIN)

it is NP-hard to decide if a given UG instance

- val $\geq 1-\delta$ "99%"
- val $\leq \delta$

where are <u>hard</u> instances of CSPs? of UG CSPs?

Random instances ?

→ Feige R3SAT conj – hard to decide between

- random instances
- planted instances.

Thm: UG are easy on expander graphs (in particular, on random instances).

[Arora-Khot-Kolla-Steurer-Tulsiani-Vishnoi 2008]

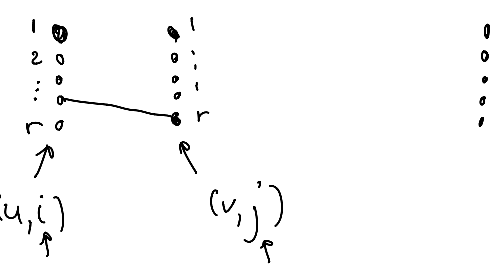
Let G be the constraint graph. $\lambda_2 = \lambda_2(G)$ (normalized)

$\exists$ alg s.t. if val $\geq 1-\varepsilon$

alg finds an assign w val $1 - \frac{\varepsilon}{(1-\lambda_2)} \cdot \lg \frac{\varepsilon}{1-\lambda_2}$.

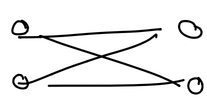
G'

label-extended graph



G' is an r-lift of G.

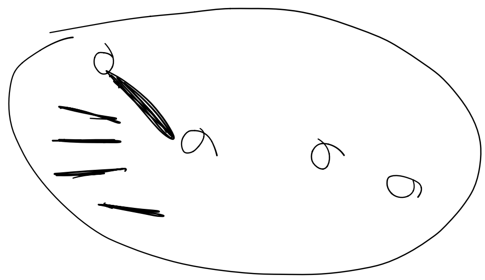
G

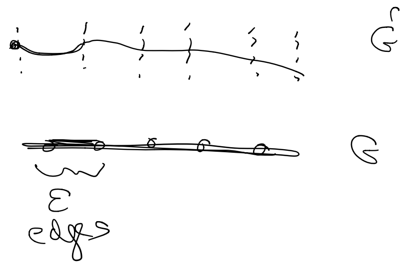


Claim: an assignment w $val \geq 1-\varepsilon$ gives a cut in G' with ε frac of edges of S leaving S.

$S, \bar{S}$

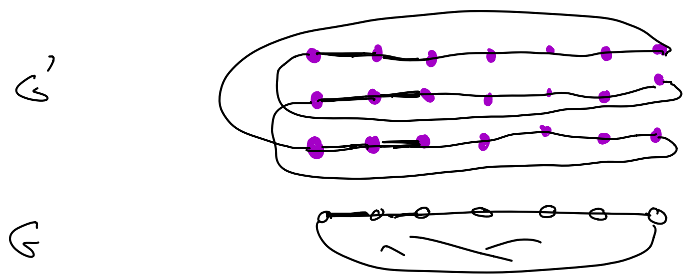
Proof:





Observe: if G d-reg then G' is also d-reg (always true for r-lift)

Example: G is a k-cycle.



is a 3k - cycle.

Proof: Suppose G UG instance, let $a : V \to [r]$ assign

$$\Pr_{u \sim_G v} \left[(a(u), a(v)) \in \Pi^{*}_{uv} \right] \geq 1 - \cancel{\varepsilon}$$

$\Pi_{uv} \subseteq [r] \times [r]$

Recal $V(G') = V \times [r]$

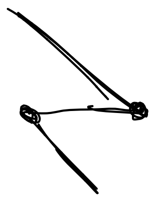
Let $S = \{ (v, a(v)) : v \in V_G \}$.

Let G^{*} be a perturbed UG instance st. a has $val = 1$.

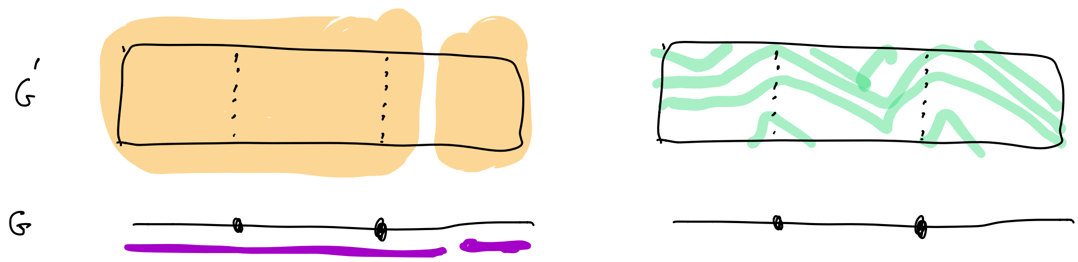
Clearly S is disconnected in $(G^{*})'$ (the label extended graph of G^{*})

The only edges leaving S in G' are above edges E^{*} = edges falsified by a.

so $|E(S,\bar{S})| \leq 2\varepsilon|S|$.



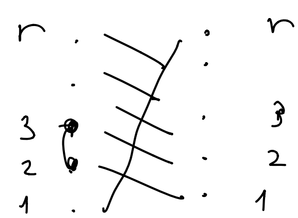
Converse: Given a UG instance G.
Let G' be the L.E. graph. A sparse cut in G', $(S,\bar{S})$ s.t. S intersects each fiber once $\rightsquigarrow$ good assignment to G.



Special case of UG. 2-LIN(r).

$x_u - x_v = c_{uv}$ $\forall$ edge $u \sim v$

3+1 7+1



Claim: Let $a: V \to [r]$

Let $a' = a+1 \mod r$

$\mathrm{val}(a) = \mathrm{val}(a')$

$S_a = \{ (v, a(v)) : v \in V_G \}$

$S_a \quad S_{a+1} \quad S_{a+2} \quad \cdots \quad S_{a+r-1}$

ALG: Given G expanding 2-LIN(r) instance.

- Construct G'.
- Find r top eigenvectors for $A_{G'}$ $y_1, \ldots, y_r$

→ • Find a sparse cut w 1 vertex per fiber in the span of $y_1 .. y_r$. $\{ \sum_{i=1}^{r} \alpha_i y_i : \alpha_i \in \text{"}\varepsilon\text{-net"} \}$

<u>Analysis</u>: Define $z_0 \cdots z_{r-1}$ indicator for $S_{a+i \bmod r}$

Claim: z_i are eigenvectors for $G^{*'}$ w eval 1.

of course, lin. ind. (and orthog)

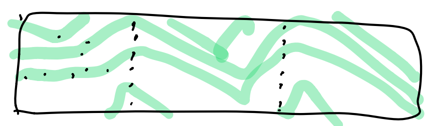
also, their Rayleigh quot is $\geq 1-2\varepsilon$

→ G' has at least r eigenvalues above $1-2\varepsilon$.

Let f be an eigenvector of $G^{*'}$ $f \perp z_1 .. z_r$.

$f^t A_{G^{*'}} f =$

can write $f = f_1 + f_2 + .. + f_r$



zero outside S_a zero outside S_{a+1}

RQ of f is normalized sum of RQ's of $f_i \leq \lambda_2(G)$.

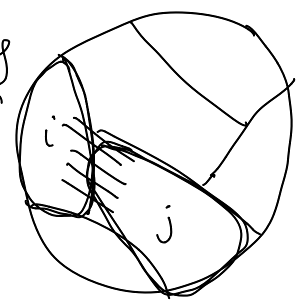
$\sum_{i=1}^{r} \alpha_i z_i$

(How to solve k-LIN on instances "splittable to expanders")

<u>Regularity Lemmas</u>

$\forall$ $G = (V, E)$ $\forall \varepsilon$ $\exists$ ε-reg partition

$\pm \varepsilon \cdot n^2$ edges



(Szemerédi reg.) tower$(\frac{1}{\varepsilon^5})$

p_{ij}

$$2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}\Big\}\frac{1}{\varepsilon^5} \qquad \exp\left(\frac{1}{\varepsilon^2}\right)$$

useful for CSPs on dense instances.

$\rightsquigarrow$ Reg lemma for "splittable tuples"