Sparsest Cut — Leighton-Rao LP relaxation

$G = (V, E)$ $\quad$ M normalized adj matrix $\quad$ (G can be k-regular)

$$\phi(G) = \min_{\substack{x \in \{0,1\}^n \\ x \neq \bar{0}, \bar{1}}} \frac{\frac{1}{k} \sum_{i \sim j} |x_i - x_j|}{\frac{1}{n} \sum_{i,j} |x_i - x_j|} = \min_{\underset{\sim}{x}} \frac{\sum_{i,j} M(i,j) |x_i - x_j|}{\frac{1}{n} \sum_{i,j} |x_i - x_j|} \quad \leftarrow G \quad \nwarrow \cancel{J} \; H$$

i,j $\quad$ d_{ij} supposed to be $|x_i - x_j|$

$\quad\quad |1_S(i) - 1_S(j)|$

Consider:

$$\min_{d \text{ is a metric}} \frac{\sum_{i,j} M(i,j) \, d_{i,j}}{\sum_{i,j} d_{ij}}$$

d needs to be a <u>metric</u> on n points.

<u>Def</u>: A metric on a finite set V, $|V| = n$, is given by $d : V \times V \to \mathbb{R}_{\geq 0}$

- $d(u,u) = 0 \quad \forall u \in V$
- $d(u,v) = d(v,u)$
- $d(u,v) \geq 0 \quad \forall u, v$
- $d(u,v) + d(v,w) \geq d(u,w)$

<u>Examples</u>:
- Euclidean distance $\quad \ell_2$ - metric

$\quad \ell_1$ - metric

[map $\varphi : V \to \mathbb{R}^D$, $\quad \mathrm{dist}(u,v) = \|\varphi(u) - \varphi(v)\|_1$.]

- Shortest-path-in-a-graph metric
- cut-metric: Fix graph G, cut $(S, \bar{S})$

$$\mathrm{dist}(u,v) = |1_S(u) - 1_S(v)|.$$

Since cut metrics are metrics

$$\min_{\substack{d \text{ non-trivial} \\ \text{metrics}}} \frac{\sum M(i,j)\, d_{ij}}{\frac{1}{n}\sum d_{ij}} \leq \phi(G) \leq \underbrace{O(\log n)}_{\text{apx-factor}} \cdot \min_{\substack{d \text{ non-trivial} \\ \text{metrics}}} \frac{\sum M(i,j)\, d_{ij}}{\frac{1}{n}\sum d_{ij}}$$

we will see

The cut cone

Suppose S, T sets, d_S cut metric

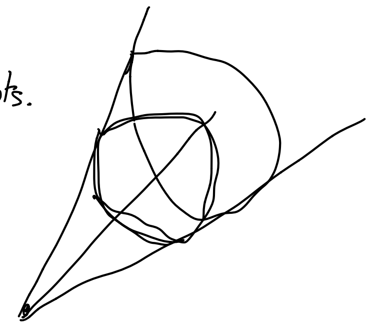
d_T cut metric.

$$d' = \alpha \cdot d_S + \beta d_T \qquad \begin{matrix}(\alpha > 0) \\ (\beta > 0)\end{matrix}$$

We think of a n-pt metric as $d \in \mathbb{R}_{\geq 0}^{\binom{n}{2}}$

The set of all metrics is convex (called a cone)

Def CUT_n = conv. hull of all cut metrics on n pts.



Thm: The cut cone $\equiv$ all ℓ_1-metrics.

Proof: $CUT_n \subseteq \ell_1$-metrics

clearly a cut metric is an ℓ_1-metric

$v \in S \rightsquigarrow 1 \in \mathbb{R}$

$v \notin S \rightsquigarrow 0 \in \mathbb{R}.$

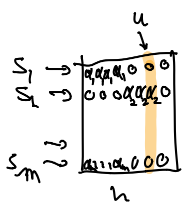
suppose $d = \sum_S \alpha_S \underbrace{d_S}$ $\qquad (\alpha_S > 0)$

cut metric $(S, \bar{S})$.

suppose we have $S_1 \ldots S_m$

we map $V \to \mathbb{R}^m$. $\quad (|V| = n)$

$$\varphi(v)_i = \begin{cases} 1 & S_i \ni v \\ 0 & S_i \not\ni v \end{cases}$$



$\forall\, u,v \in V \qquad d(u,v) \stackrel{?}{=} \|\varphi(u)-\varphi(v)\|_1$

$$\|\varphi(u)-\varphi(v)\|_1 = \sum_{i=1}^{m} |\varphi(u)_i - \varphi(v)_i| =$$

$$= \sum_{i=1}^{m} \alpha_{S_i} \, |1\!\!1_{S_i}(u) - 1\!\!1_{S_i}(v)|$$

$$= \sum_{i=1}^{m} \alpha_{S_i} \, d_{S_i}(u,v) = d(u,v).$$

ℓ_1- metrics $\subseteq$ cut_n :

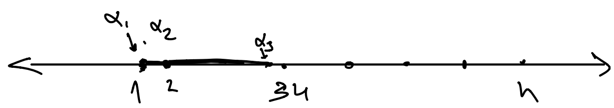
step 1 : Suppose d is an ℓ_1 metric.

$d = \sum_{i=1}^{m} d_i$ where d_i is 1-dim ℓ_1-metric.

$\exists\, \varphi: V \to \mathbb{R}^m,\ d(u,v) = \sum_{i=1}^{m} |\varphi(u)_i - \varphi(v)_i|$ (take $\varphi_i(u) = \varphi(u)_i$ & def $d_i(u,v) = |\varphi_i(u) - \varphi_i(v)|$.)

Step 2: suppose d is an ℓ_1-metric 1-dimensional.

$\exists \quad \varphi: V \to \mathbb{R}$



let $\{\alpha_1, \dots, \alpha_n\}$ = values of φ.

define $n-1$ cuts $S_i = \{ v \in V \mid \varphi(v) \le \alpha_i \}$

want to find β s.t. $\quad d = \sum_{i=1}^{n-1} \beta_i \cdot d_{S_i}$ where $\beta_i = \alpha_{i+1} - \alpha_i$

Remark: Suppose d an ℓ_1- metric $\left(= \sum d_S \cdot \beta_S\right)$ $\quad \dfrac{\sum_{i,j} M(i,j)\, d_{i,j}}{\frac{1}{n}\sum_{i,j} d_{ij}} \le \alpha$

then we can alg. find a cut with value α.

$$\sum M(i,j)\, d_{ij} - \alpha \tfrac{1}{n} \sum_{i,j} d_{ij} = 0$$

$$\sum_S \beta_S \left[\sum_{ij} M(i,j)\, d_S(i,j) - \tfrac{\alpha}{n} \sum d_S(i,j) \right] \le 0$$

$\to \exists S$ s.t. ≤ 0 cut $S, \bar{S}$ has value $\leq \alpha$.

Alg we can run an LP to find a metric d with smallest sparsity.

Metric embedding Given $d: V \times V \to \mathbb{R}_{\geq 0}$ a metric

we want $\varphi: V \to \mathbb{R}^m$ (some $m \in \mathbb{N}$)

s.t. $\frac{1}{\log n} \|\varphi(u) - \varphi(v)\|_1 \leq d(u,v) \leq \|\varphi(u) - \varphi(v)\|_1$

the distortion

(let $d_\varphi(u,v) = \|\varphi(u) - \varphi(v)\|_1$).

Thm: Every metric d embeds into ℓ_1 w distortion $O(\log n)$!
[B, LLR] dim of embedding $\leq (\log n)^2$.

$$\alpha = \frac{\sum M(i,j)\, d(i,j)}{\frac{1}{n}\sum_{i,j} d(i,j)} \geq \frac{\sum M(i,j)\, d_\varphi(i,j)}{\sum d_\varphi(i,j) \cdot \log n}$$

LP optimization value

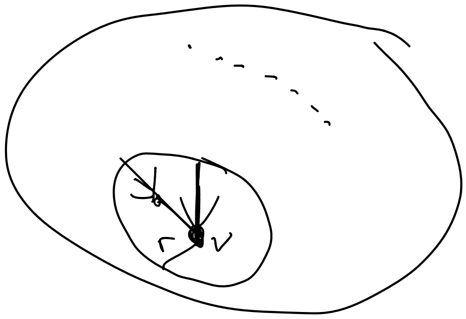
$\to$ d_φ has value $\alpha \cdot \log n$

$\to$ can find a cut $S, \bar{S}$ from d_φ w value $\alpha \cdot \log n$.

Since the best cut has sparsity $\geq \alpha$, our alg finds a $\log n$ - approximation.

How good is this LR-approx for sparsest cut?

$\log n$ is correct, as can be seen by G an expander:



G $\quad$ k-regular

$r \leq \log_k n - 1$

s.t. $k^r < \frac{n}{2}$

$$\sum_{i,j} d_G(i,j) \approx r^2 \cdot n^2$$

Spectral alg & LR-alg

mapping vectors into ℓ_2^2 $\qquad$ mapping $d \to \ell_1$

$\sum M(i,j) \cdot (x_i - x_j)^2$ $\qquad$ $\sum M_{ij} |x_i - x_j| \leftarrow \ell_1$

. ARV showed a semi-def. prog. $\sqrt{\log n}$

$\forall u,v \quad 0 \leq d_{u,v}$

$d_{uw} \leq d_{uv} + d_{vw}$

normalization $\to$ $\sum_{u,v} d_{uv} = 1$

$\max \sum M(u,v)\, d(u,v)$