Sparsest cut — Leighton-Rao LP relaxation

\[ G = (V, E) \]

\[ M \text{ adj matrix} \]

\[ \phi(G) = \min_{x \in \mathbb{R}^{|V|}} \frac{1}{k} \sum_{i,j} |x_i - x_j| \]

\[ \text{s.t. } \sum_{i,j} M(i,j) |x_i - x_j| \leq 1 \]

\[ G \] can be \( k \)-regular.

**Example:** Euclidean distance \( l_2 \)-metric

\[ \| \mathbf{u} - \mathbf{v} \|_2 \]

**Example:** Shortest-path-in-a-graph metric

**Example:** Cut metric: Fix graph \( G \), cut \((S, \overline{S})\)

\[ \text{dist}(u, v) = |\mathbf{1}_S(u) - \mathbf{1}_S(v)| \]

Since cut metrics are metrics.
The cut cone

Suppose \( S, T \) sets, \( d_S \) cut metric
\( d_T \) cut metric.

\[
d' = \alpha \cdot d_S + \beta d_T \\
\text{(\( \alpha > 0 \)) \text{(\( \beta > 0 \))}
\]

We think of \( a \) metric as \( d \in \mathbb{R}^{(h)}_{>0} \)

The set of all metrics is convex (called a cone)

\[
\text{Def } \text{CUT}_n = \text{conv. hull of all cut metrics on } n \text{ pts.}
\]

Thus: The cut cone \( \equiv \) all \( l_1 \)-metrics.

Proof:
\( \text{CUT}_n \subseteq \text{all } l_1 \)-metrics

Clearly a cut metric is an \( l_1 \)-metric
ves \( \sim 1 \in \mathbb{R} \)
\( \forall \text{ } S \sim a \in \mathbb{R}. \)

Suppose \( d = \sum_{S} d_S \) \( S \) cut metric \((S,S)\).

Suppose we have \( S_1 \ldots S_m \)

we map \( V \to \mathbb{R}^m, (|V|=n) \)

\[
\chi_i(V) = \begin{cases} 
1 & \text{if } S_i \in V \\
0 & \text{if } S_i \notin V
\end{cases}
\]

we will see \( \min \sum_{S} d_S \) a\( \text{p.x. factor.} \)
\[ \forall u, v \in V \quad d(u, v) = \| \phi(u) - \phi(v) \|_1 \]

\[ \| \phi(u) - \phi(v) \|_1 = \sum_{i=1}^{m} |\phi_i(u) - \phi_i(v)| = \]

\[ = \sum_{i=1}^{m} \alpha_{s_i} |s_i(u) - s_i(v)| = \sum_{i=1}^{m} \alpha_{s_i} d_{s_i}(v) = d(u, v). \]

**\( \ell_1 \)-metrics \subseteq \text{cut}_n:**

**Step 1:** Suppose \( d \) is an \( \ell_1 \)-metric.

\[ d = \sum_{i=1}^{m} d_i, \quad \text{where } d_i \text{ is } 1\text{-dim } \ell_1\text{-metric.} \]

\[ \exists \phi : V \to \mathbb{R}^m, \quad d(u, v) = \sum_{i=1}^{m} |\phi_i(u) - \phi_i(v)| \quad \text{(take } \phi_i(u) = \phi_i(v) \text{)} \]

\[ \& \text{def } d_i(u, v) = |\phi_i(u) - \phi_i(v)|. \]

**Step 2:** Suppose \( d \) is an \( \ell_1 \)-metric \( 1 \)-dimensional.

\[ \exists \phi : V \to \mathbb{R} \]

Let \( \alpha_1, \ldots, \alpha_n \) = values of \( \phi \).

Define \( n-1 \) cuts \( S_i = \{ v \in V \mid \phi(v) \leq \alpha_i \} \)

\[ \text{want to find } p \text{ st. } d = \sum_{i=1}^{n-1} \beta_i \cdot d_{S_i} \quad \text{where } \beta_i = \alpha_{i+1} - \alpha_i. \]

**Remark:** Suppose \( d \) an \( \ell_1 \)-metric

\[ \sum_{i,j} M(i,j) d_{i,j} \leq \lambda \]

\[ \sum_{i,j} d_{i,j} \cdot \beta \]

Then we can algo. find a cut with value \( \lambda \).

\[ \sum_{i,j} M(i,j) d_{i,j} - \sum_{i,j} d_{i,j} \leq 0 \]

\[ \sum_{i,j} \beta_{s_i} \left( \sum_{i,j} M(i,j) d_{i,j} - \sum_{i,j} d_{i,j} \right) \leq 0 \]
∀ S s.t. \( \exists \) cut \( S, \bar{S} \) has value \( \leq \alpha \).

Alg: we can run an LP to find a metric \( \delta \) with smallest sparsity.

Metric embedding

Given \( d : V \times V \to \mathbb{R}_{\geq 0} \), a metric

we want \( \psi : V \to \mathbb{R}^m \) (some \( m \in \mathbb{N} \))

s.t. \( \frac{1}{\log n} \| \psi(u) - \psi(v) \|_1 \leq d(u,v) \leq \| \psi(u) - \psi(v) \|_1 \)  

(let \( d_\psi(u,v) = \| \psi(u) - \psi(v) \|_1 \)).

Thm: Every metric \( d \) embeds into \( L_1 \) with distortion \( o(\log n) \).

\([B, LLR]\) dim of embedding is \( (\log n)^2 \).

\[ \begin{align*}
\alpha &= \frac{\sum_{i,j} M(i,j) d(i,j)}{\sum_{i,j} d(i,j)} \\
&\geq \frac{\sum_{i,j} M(i,j) d_\psi(i,j)}{\sum_{i,j} d_\psi(i,j)} \cdot \log n
\end{align*} \]

\( \Rightarrow \) \( d_\psi \) has value \( \alpha \cdot \log n \)

\( \Rightarrow \) can find a cut \( \delta, \bar{\delta} \) from \( d_\psi \) w value \( \alpha \cdot \log n \).

Since the best cut has sparsity \( \geq \alpha \), our alg finds a \( \log n \)-approximation.

How good is this LR-approx for sparsest cut?

\( \log n \) is correct, as can be seen by \( G \) an expander:
\[ G \quad \text{k-regular} \]
\[ r \leq \log_k n - 1 \]
\[ \text{s.t.} \quad k^r < \frac{n}{2} \]

\[ \sum_{i,j} d(i,j) \approx r^2 n^2 \]

Spectral alg & LR-alg

* mapping vectors into \( l^2_2 \)
* mapping \( d \rightarrow l \)

\[ \sum_{i,j} (x_i - x_j)^2 \]

\[ \sum_{i,j} |x_i - x_j| \leftarrow l \]

ARV showed a semi-def. prog. \( \sqrt{\log n} \)

\[ \forall u,v \quad d_{u,v} \geq 0 \]

\[ d_{u,v} \leq d_{u,w} + d_{w,v} \]

normalization \( \bigoplus d_{u,v} = 1 \)

\[ \max_x \sum_{u,v} E(M(u,v) d_{u,v}) \]