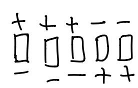
Mixing in Markov Chains : spaces with exp num of configurations

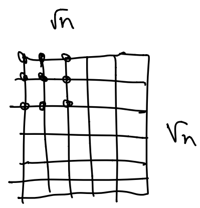
Card shuffling. 52 cards. $52!$ orders $\sim \underline{10^{77}}$

- random transposition $\underline{(i\,j)}$ 100
- top to random $\underline{(1\,i)}$ 300
- "rifle shuffle" -- 8

$\mathcal{D}$ – 20% far from uniform over G
distr over permutations

Ising model (spin-glass)





x – config $H(x)$ – energy of config

Gibbs distribution $p(x) \sim e^{-\beta H(x)}$ "partition function"

Glauber dynamics : choose vertex at random, resample.

MCMC method

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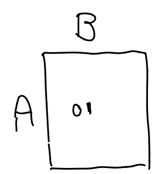
sampling $\sim$ approx. counting.

Given G bipartite. Count # perfect matchings in G.

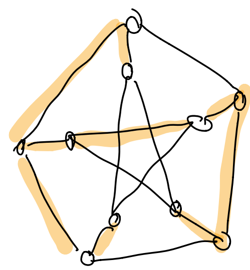
$\Longleftrightarrow$ calc permanent of matrix of G.



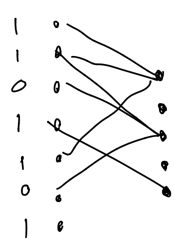
#P – complete.

Thm [JSV]: There is a fully poly apx scheme for counting perf. matchings.

Example: Sampling min. spanning trees in a graph.



Example: satisfying assignments in a CSP (k-SAT)

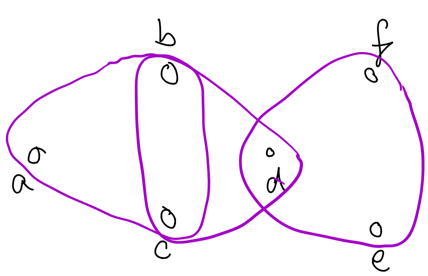


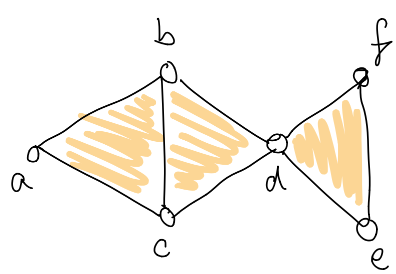
Simplicial Complex X is a hypergraph s.t. $S \in X$, $S' \subseteq S$ $\rightarrow S' \in X$

Graph V, E = collection of size-2 subsets of V

Hypergraph V F = collection of subsets of V

3-uniform h.g.: all hyperedges have 3 elements.

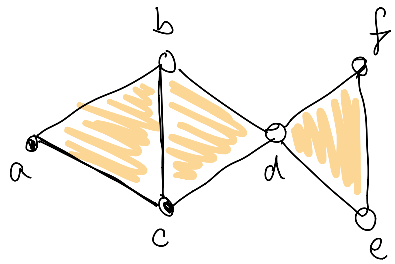




High Dimensional Expansion

- Random Walks
- expanders $\approx$ golden standard ?
- Links def.

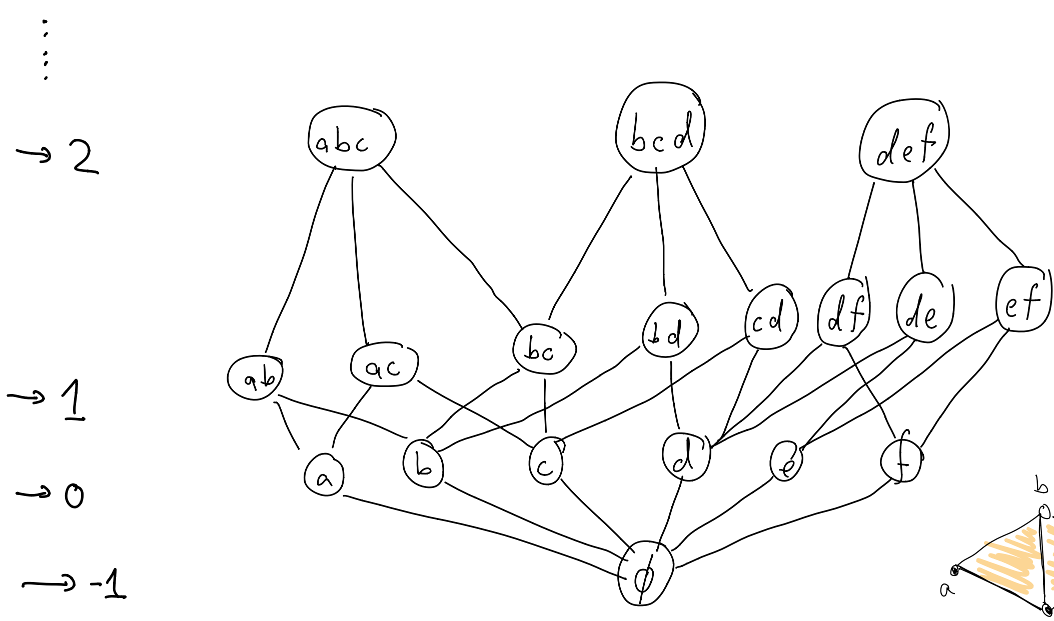
vertex — edge — vertex
edge — vertex — edge
triangle — edge — triangle
edge — triangle — edge

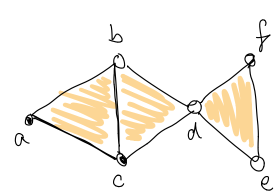


"2-dim s.c."

<u>Def</u>: X simp. comp. $X(0)$ – vertices
$X(1)$ – edges
$X(i)$ - i-dim faces $= \{s \in X : |s| = i+1\}$

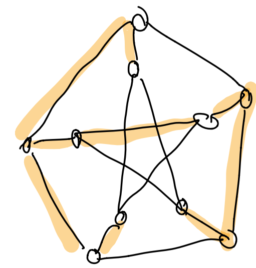
X - is d-dim if $X(d+1) = \phi$.





$|E| = m$

Given a graph $G = ([n], E)$, we construct a (n-2)-dimensional simp. complex. X



$|E| = 15$
$|V| = 10$

$X(0) = E$

$X(i) = \{ S \subset X(0) \mid |S| = i+1 \ \& \ S \text{ can be completed to spanning tree} \}$

$\underbrace{X(n-2)}_{\text{sets of } n-1 \text{ elements}} = \{ S \mid S \text{ is a spanning tree} \}$

G graph $\quad M$ - normalized adj mat, w ev $\lambda_1, \dots, \lambda_n$

G - γ- two-sided expander if $\lambda_2, |\lambda_n| \le \gamma$

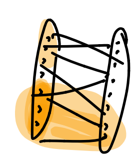
$\forall\, i > 1 \quad -\gamma \le \lambda_i \le \gamma$

$$\left\| M - \tfrac{1}{n} \cdot J \right\|_2 \le \gamma$$

EML $\quad S, T$

$$\|A\| = \sup_{f \neq 0} \frac{\|Af\|}{\|f\|} = \lambda_{max}$$

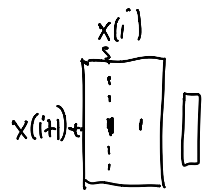
γ - one-sided expander $\quad \lambda_2 \le \gamma$

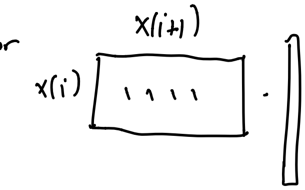


$$M - \tfrac{1}{n} J \preceq \gamma I$$

Cheeger's $\quad S, \bar{S}$

UD & DU walks

Up operator : 

Down operator 

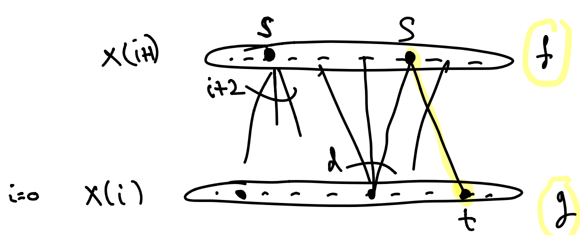
$\langle f, g \rangle_i = \mathbb{E}_{x \in X(i)} f(x)\, g(x)$

$f, g : X(i) \longrightarrow \mathbb{R}$

$\langle f, Ug \rangle_{i+1} = \langle Df, g \rangle_i$

$g : X(i)$
$f : X(i+1)$

$\langle f, Ug \rangle_{i+1} = \langle Df, g \rangle_i$



$$\begin{array}{c|cc} & A & B \\ \hline A & 0 & T \\ B & T^t & 0 \end{array}$$

$$\langle f, Ug \rangle = \mathbb{E}_{s \in X(i+1)} \left[f(s) \cdot Ug(s) \right] = \boxed{\mathbb{E}_{\substack{s \supset t \\ X(i+1) \quad X(i)}} f(s) \cdot g(t)} = \mathbb{E}_{t}\, g(t) \cdot \mathbb{E}_{\substack{s \supset t \\ s \in X(i+1)}} f(s)$$

since $Ug(s) := \mathbb{E}_{\substack{t \subset s \\ t \in X(i)}} g(t)$ $\qquad = \langle Df, g \rangle_i$

UD_i "downup" vs DU_i "up down"

for $i=0$ $\quad UD_i = \frac{1}{n} J$

$DU_i = \underbrace{\tfrac{1}{2} Id}_{} + \frac{1}{2} M_G$ (where G is the 1-skeleton)

We define L_i^+ non-lazy upper random walk, by the process:

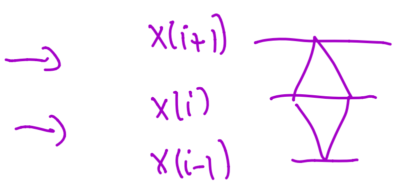
- start at $s \in X(i)$
- choose random $t \supset s$ $t \in X(i+1)$ uniformly
- choose $s' \subset t$ $s' \in X(i)$ uniformly conditioned on $s' \neq s$.

$$DU_i = \frac{1}{i+2} \cdot Id + \frac{i+1}{i+2} \cdot L_i^+ . \quad \left(\begin{array}{l} \text{for } i=0 \\ L_i^+ = M_G \end{array} \right)$$

G is a γ-expander iff $\| L_i^+ - UD \| \leq \gamma$

<u>Def:</u> A simp. complex $X^{(d)}$ is a RW-2-sided HDX w. param γ if $\forall\ 0 \leq i \leq d-1$ $\| L_i^+ - UD_i \| \leq \gamma$.

$$\begin{array}{c} DU_i \\ \| \\ \frac{1}{2}I + \frac{1}{2}L_i^{1} \end{array}$$



$L_i^+ - UD_i \preccurlyeq \gamma I$ (one-sided)

Let $\Delta_n^{(k)}$ be the complete k-dim s.comp on n vertices.

<u>Lemma:</u> Let $X^{(k)}$ be a γ-high dim expander (two sided)

$\forall\ 0 \leq i < k$ $\quad \lambda_2\left(L_i^+(X) \right) = \lambda_2\left(L_i^+(\Delta) \right) + o(\gamma)$

& $\quad \lambda_2\left(L_i^+(X) \right) = \frac{i+1}{i+2} + o(\gamma)$.

Conclusion: γ-HDX "approximate" the complete complex

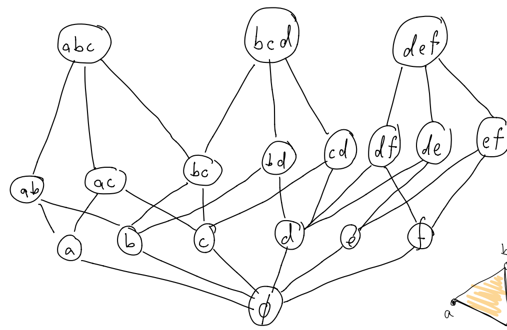
<u>Link-Expansion</u>

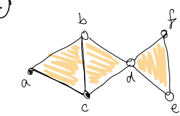
Def Let X be k-dim.

Fix $s \in X(i)$

$\mathrm{Link}(s) = \{ t \setminus s \mid s \subset t \in X \}$

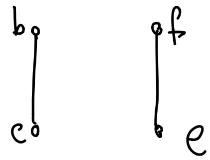
$\dim (k-i-1)$ dimensional





Link (a) =

Link(d) =



$\forall s \in X(i)$ Let G_s be the $\overbrace{\text{1-skeleton}}^{\text{graph}}$ of $\mathrm{Link}(s)$.

Def (link expander) A k-dim s.complex X is a β-link-HDX if $\forall\, s \in X(i)$, $i < k-1$ G_s is a β- ~~two-sided~~ expander

two-sided $\lambda_2, |\lambda_n| \le \beta$.

one-sided $\lambda_2 \le \beta$

Thm: A s.c. X is a γ- RW-HDX $\Leftrightarrow$ $O_k(\gamma)$ - link HDX. (two-sided)

Thm: If X γ- one-sided link HDX $\Rightarrow$ $O(\gamma)$ - RW - HDX.

Proof:

Do there exist sparse 2-dim HDXs ?