Expanders - exercise 1
Instructor: Irit Dinur
Due: Monday, November 16, 2020

Instructions: Please join into small groups, work together and submit together. Ideally I am hoping for groups of 2-4 students. Please type your solutions using LaTeX. Please submit your file to https://www.dropbox.com/request/u8qBSy3guCcr1hymJY1o

1 Super concentrators

- Let \( G \) be a graph with special vertex sets \( I, O \subset V \). Suppose that for every \( k \) and every \( S \subseteq I \) and \( T \subseteq O \) with \( |S| = |T| = k \), there are no \( k-1 \) vertices whose removal disconnects every \( s \in S \) from every \( t \in T \). Show that \( G \) is a super-concentrator.

- A linear circuit is a circuit where every gate computes a linear function of the inputs. Let \( A \) be a super-regular matrix. Show that the graph of a linear circuit computing the transformation \( x \mapsto Ax \) is a super concentrator.
  (Recall that a super-regular matrix is a square matrix such that any square sub-matrix has full rank).

2 Good error correcting codes

Show that there exists some \( \epsilon_0 > 0 \) such that for every \( n \) there is a code with relative rate at least \( \epsilon_0 \) and relative distance at least \( \epsilon_0 \). Hint: For a fixed distance, say \( \delta = 1/4 \), construct a code with distance \( \delta \) by adding codewords greedily, and show that you can squeeze in sufficiently many codewords.

3 Amplification of soundness in randomized algorithms

An \((n, n, d)\)-graph is a bipartite graph with \( n \) vertices on each side and the degree of each left vertex is \( d \). The graph has property \((exp)\) if for every subset \( S \) of left vertices with size at most \( \frac{2}{d} n \), the set of neighbors of \( S \) has size at least \( \frac{4}{3}|S| \).

- Show that a random \((n, n, d)\) graph obtained by having each left vertex choose \( d \) random vertices has property \((exp)\) with probability greater than 3/4. You may assume that \( n, d \) are large enough.

- Deduce that for every set \( B \) of right vertices of cardinality less than \( n/2 \), if \( S \) is a set of left vertices such that \( \Gamma(S) \subseteq B \), then \( |S| < \frac{2}{3}n \).

- Let \( G \) be an \((n, n, d)\) graph with property \((exp)\), such that \( n = 2^k \). Suppose \( A(\cdot, \cdot) \) is a randomized algorithm for deciding a language \( L \) such that \( A \) uses \( k \) bits of randomness and has one-sided error of 1/2. Show, using \( G \), an algorithm that uses the same number of random bits \( k \) but has soundness error of \( 2/d \).