# Expanders - exercise 1

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Due: Monday, November 16, 2020

**Instructions:** Please join into small groups, work together and submit together. Ideally I am hoping for groups of 2-4 students. Please type your solutions using LaTeX. Please submit your file to `https://www.dropbox.com/request/u8qBSy3guCcr1hymJY1o`

## 1 Super concentrators

- Let $G$ be a graph with special vertex sets $I, O \subset V$. Suppose that for every $k$ and every $S \subseteq I$ and $T \subseteq O$ with $|S| = |T| = k$, there are no $k-1$ vertices whose removal disconnects every $s \in S$ from every $t \in T$. Show that $G$ is a super-concentrator.
- A linear circuit is a circuit where every gate computes a linear function of the inputs. Let $A$ be a super-regular matrix. Show that the graph of a linear circuit computing the transformation $x \mapsto Ax$ is a super concentrator.
  (Recall that a super-regular matrix is a square matrix such that any square sub-matrix has full rank).

## 2 Good error correcting codes

Show that there exists some $\epsilon_0 > 0$ such that for every $n$ there is a code with relative rate at least $\epsilon_0$ and relative distance at least $\epsilon_0$. Hint: For a fixed distance, say $\delta = 1/4$, construct a code with distance $\delta$ by adding codewords greedily, and show that you can squeeze in sufficiently many codewords.

## 3 Amplification of soundness in randomized algorithms

An $(n, n, d)$-graph is a bipartite graph with $n$ vertices on each side and the degree of each left vertex is $d$. The graph has property ($exp$) if for every subset $S$ of left vertices with size at most $\frac{2}{d}n$, the set of neighbors of $S$ has size at least $\frac{d}{4}|S|$.

- Show that a random $(n, n, d)$ graph obtained by having each left vertex choose $d$ random vertices has property ($exp$) with probability greater than 3/4. You may assume that $n, d$ are large enough.
- Deduce that for every set $B$ of right vertices of cardinality less than $n/2$, if $S$ is a set of left vertices such that $\Gamma(S) \subseteq B$, then $|S| < \frac{2}{d}n$.
- Let $G$ be an $(n, n, d)$ graph with property ($exp$), such that $n = 2^k$. Suppose $A(\cdot, \cdot)$ is a randomized algorithm for deciding a language $L$ such that $A$ uses $k$ bits of randomness and has one-sided error of 1/2. Show, using $G$, an algorithm that uses the same number of random bits $k$ but has soundness error of $2/d$.