Expanders - exercise 3

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Due: Tuesday, January 5, 2021

Instructions: Please work in small groups and submit together. Ideally I am hoping for groups of 2-4 students. Please type your solutions using LaTeX. Please submit your file to `https://www.dropbox.com/request/3mpQL6urdGufoDor1lVw`

1 Spectrum and Combinatorial Properties

Let G be a d-regular graph on n vertices. Let M be its normalized adjacency matrix and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its eigenvalues.

- Assume G is connected. Show that G is bipartite iff $\lambda_n = -1$.
- Show that if G is bipartite then whenever λ is an eigenvalue, also $-\lambda$ is an eigenvalue.
- In class we proved the expander mixing lemma and used $\lambda = max(-\lambda_n, \lambda_2)$ to bound the error term. Show a graph and subsets S, T that demonstrate why we cannot use λ_2 instead.
- Bonus: formulate and prove an expander mixing lemma for bipartite graphs.

2 Cayley graphs of Abelian groups

1. Suppose that G_1, G_2 are Abelian groups, and let $S_i \subset G_i$ be generating sets closed under inverses, for $i = 1, 2$. Prove that

$$Cay(G_1, S_1) \otimes Cay(G_2, S_2) = Cay(G_1 \times G_2, S_1 \times S_2)$$

   where the tensor product of graphs is defined as per the previous exercise (by tensoring the adjacency matrices), and where $G_1 \times G_2$ is the group that is the direct product of the two groups.

2. We will prove that Cayley graphs of Abelian groups can never be families of bounded-degree expanders. Suppose G is an Abelian group and S a set of generators closed under inverses.

   (a) For any positive integer r, let

$$Ball_r(v) = \{u \in V \mid dist(u, v) \leq r\}$$

   where $dist(u, v)$ is the length of the shortest path in G from u to v.

   (b) Prove a lower bound on the diameter of $Cay(G, S)$, and deduce the result by contrasting with the diameter of an expander graph.

3 Error correcting codes from expanders

1. Suppose that $G = (L, R, E)$ is an $(n, m, D, \gamma, D(1-\varepsilon))$ graph as defined in Lecture 14. Show that the code defined below has relative distance $2\gamma(1-\varepsilon)$,

$$C(G) = \{z \in \{0,1\}^n \mid \forall r \in R, \sum_{i \sim r} z_i = 0 \mod 2\}$$

2. Describe a string $y \in \{0,1\}^n$ with $dist(y, C(G)) \leq \gamma n$ such that the decoding algorithm (seen in class) increases the distance of y from the code in an intermediate step.

3. Suppose that $G = (L, R, E)$ is an $(n, \frac{3}{4}n, D, \gamma, D(1-\varepsilon))$ graph, and suppose that after deleting a vertex $r_0 \in R$ from G, the resulting graph $G' = (L, R \setminus \{r_0\}, E')$ still has the property that every set $S \subset L$ with $|S| \leq \gamma n$ has at least $(1-2\varepsilon)D|S|$ unique neighbors.

   (a) Prove that the code $C(G')$ has distance at least γn.

   (b) If G is chosen at random, then whp the constraint defined by r_0 is linearly independent from the other constraints with high probability, so $C(G') \supsetneq C(G)$. Choose any $y \in C(G') \setminus C(G)$. How many constraints of $C(G)$ does y violate? Give a lower bound on the distance of y from $C(G)$. How does this fit with the decoding algorithm and its analysis?