Expanders - exercise 4

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Due: Tuesday, January 19, 2021

Instructions: Please work in small groups and submit together. Ideally I am hoping for groups of 2-4 students. Please type your solutions using LaTeX. Please submit your file to `https://www.dropbox.com/request/3mpQL6urdGufoDor1lVw`

1 The lines-points graph

In this exercise we will find the spectral decomposition of a graph in a special case which can be extended to more general"distance regular" graphs. Let $\mathbb{F}$ be a finite field with q elements and let $m > 1$ be an integer. An affine line in $\mathbb{F}^m$ is defined by two points $a, b \in \mathbb{F}^m$ by $\ell_{a,b} = \{a + tb \mid t \in \mathbb{F}\}$.

1. Let L be the set of all affine lines. Show that $|L| = q^m(q^m - 1)/q(q - 1)$.
2. Define the "lines-vs.-points" bipartite graph $G = (P = \mathbb{F}^m, L, E)$ with $(p, \ell) \in E$ iff $p \in \ell$. Let A be the $|P| \times |L|$ matrix such that $A(x, \ell) = 1$ iff $x \in \ell$ and $A(x, \ell) = 0$ otherwise.
3. Define the graph $G_L = (L, E)$ by connecting $\ell, \ell' \in L$ iff they intersect on a point. Express the adjacency matrix of G_L using A.
4. Define the graph $G_P = (\mathbb{F}^m, E)$ by connecting $x, x' \in \mathbb{F}^m$ iff they are together contained in a line. Express the adjacency matrix of G_P using A.
5. Show that $(M_L)^2 = \alpha_1 I + \alpha_2 J + \alpha_3 M_L$ for appropriate $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, where M_L is the normalized adjacency matrix of G_L, and where J is the all ones matrix and I is the identity matrix. Use this to find the eigenvalues of M_L. (A similar calculation can work for G_P).

2 Semi-direct product, Cayley groups, and the replacement product

1. Given a finite group G, $Aut(G)$ is the set of automorphisms, namely all bijections $\phi : G \to G$ that respect the group structure, namely $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$. Give an example of two automorphisms of the group $H = Z/nZ$, the cyclic group with n elements.
2. An action of a group H on another group G is a homomorphism $\phi : H \to Aut(G)$. We denote $\phi(h)[g]$ by g^h. For example, the group $H = Z/nZ$ acts on the group $G = \{0, 1\}^n$ by cyclicly shifting the coordinates of each string. Warmup: how does the element $+2 \in Z/6Z$ act on the string 010000? How about on 010101?
3. The orbit of an element $g \in G$ under the action of H is the set $\{g^h \mid h \in H\}$. What are the orbits of the two strings above (wrt the same action)?

4. Given two finite groups G, H and an action of H on G, define a new group $G \rtimes H$ as follows. The elements are the set of all pairs (g, h) such that $g \in G$ and $h \in H$. The group operation is defined to be

$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2^{h_1}, h_1 h_2)$$

Show that this is indeed a group. What happens when H acts trivially on G (i.e. $g^h = g$ for all g, h)?

5. Fix $G = (Z/2Z)^3$ and $H = Z/3Z$, and consider the group $G \rtimes H$. Let $Cay(G, \{e_1, e_2, e_3\})$ be the Hamming cube graph, and let $Cay(H, \{\pm 1\})$ be the triangle graph. Draw the replacement product of G and H and show that this is a Cayley graph of $G \rtimes H$. What are the generators?

6. Let G, H be finite groups, such that H acts on G. Let S_H, S_G be generating sets of H, G, and assume that S_G is an orbit of some element $g \in G$ under the H action, namely, $S_G = \{g^h \mid h \in H\}$, and further assume that $|H| = |S_G|$. The replacement product of $Cay(G, S_G)$ with $Cay(H, S_H)$ is a Cayley graph of the group $G \rtimes H$. What are the generators of this graph? (please prove your answer)

3 ϵ-biased sets

Let $S \subset \{0,1\}^k$ be a set of size n. Prove that the following are equivalent

1. Let A be the n by k matrix whose rows are the elements of S. The columns of A generate an error correcting code, namely, a linear subspace $C \subset \{0,1\}^n$ for which whenever $x \neq y \in C$, $\frac{1-\varepsilon}{2} \leq \frac{1}{n} dist(x, y) \leq \frac{1+\varepsilon}{2}$. Assume that $KerA = \{0\}$.

2. For every $\bar{0} \neq \alpha \in \{0,1\}^k$,

$$|\mathbb{E}_{s \in S}[\chi_\alpha(s)]| = |\mathbb{E}_{s \in S}[(-1)^{\sum_{i=1}^k s_i \alpha_i}]| \leq \varepsilon$$

3. The Cayley graph $G = Cay(\{0,1\}^k, S)$ has $\lambda(G) \leq \varepsilon$ (here $\lambda(G) = \max(\tilde{\lambda}_2, -\tilde{\lambda}_n)$ is normalized).

Such a set S is called an ε-biased set.