Homework 2 high dimensional expanders

January 10, 2023

You may work in groups of 2-3 students. Please write up your homework in $LaTeX$ and submit the pdf by 24.1.2022 to *yotamd@weizmann.ac.il*. Please feel free to approach Yotam or Irit with any questions.

1 Link expansion in Kaufman and Oppenheim's graphs

Let p be some prime. In this question all polynomials are over $\mathbb{F}_p$. This proof first appeared in [OP22].

1. Let $Y = (L, R, E)$ be the bipartite graph that is a link in the Kaufman Oppenheim complex. That is,

$$L = R = \{(\ell(x), Q(x)) \mid \ deg(\ell(x)) \leqslant 1, deg(Q(x)) \leqslant 2\}$$

and

$$(Q(x), \ell(x)) \sim (Q'(x), \ell'(x)) \Leftrightarrow Q'(x) - Q(x) = \ell(x)\ell'(x).$$

Show that two vertices $(\ell(x), Q(x)), (\ell'(x), Q'(x)) \in L$ have distance 2 if and only if there exists two linear polynomials $\ell_1(x), \ell_2(x)$ so that

$$(\ell'(x), Q'(x)) = (\ell(x), Q(x)) + (\ell_1(x), \ell_1(x)\ell_2(x)).$$

2. Let G be a group and $S \subseteq G$ be so that for all $s \in S$, $s^{-1} \in S$. The Cayley graph $Cay(G, S)$ is a graph whose vertices are the elements in G, and $a \sim b$ if $b = a \cdot s$. Show that the two-step walk of the left side of Y is isomorphic to a Cayley graph $Cay(G, S)$ where $G = \mathbb{F}_p^5$ and

$$S = \{(a, b, ac, ad + bc, bd) \mid \ a, b, c, d \in \mathbb{F}_p\}.$$

3. Let $\omega = e^{2\pi i/p} \in \mathbb{C}$. Let $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5) \in \mathbb{F}_p^5$. Show that the function $f_{\mathbf{r}} : \mathbb{F}_p^5 \to \mathbb{C}$, $f_{\mathbf{r}}(\mathbf{x}) = \omega^{\langle \mathbf{r}, \mathbf{x} \rangle}$ is an eigenfunction of the adjacency operator of $Cay(\mathbb{F}_p^5, S)$. Show that its eigenvalue is $\mathbb{E}_{s \in S}[f_{\mathbf{r}}(s)]$. Here $\langle \mathbf{r}, \mathbf{x} \rangle = \sum_{i=1}^5 r_i x_i \ (mod\ p)$.

4. Fix some $\mathbf{r} \in \mathbb{F}_p^5$. Let $h_1(c, d) = r_1 + r_3 c + r_4 d$ and $h_2(c, d) = r_2 + r_4 c + r_5 d$. Show that

$$\mathop{\mathbb{E}}_{s \in S}[f_{\mathbf{r}}(s)] = \mathop{\mathbb{E}}_{c,d}\left[\mathop{\mathbb{E}}_{a}\left[\omega^{a \cdot h_1(c,d)}\right] \cdot \mathop{\mathbb{E}}_{b}\left[\omega^{b \cdot h_2(c,d)}\right]\right].$$

Conclude that $\mathbb{E}_{s \in S}[f_{\mathbf{r}}(s)] = \mathbb{P}_{c,d}[h_1(c, d) = h_2(c, d) = 0]$.

5. Assume that $\mathbf{r} \neq (0,0,0,0,0)$. Show that $\mathbb{P}_{c,d}\left[h_1(c,d) = h_2(c,d) = 0\right] \leqslant \frac{1}{p}$. Use this to bound the non-trivial eigenvalues of the two-step walk of the left side of Y, and conclude that Y is a $\frac{1}{\sqrt{p}}$-one-sided spectral expander.

2 Coboundary and cosystolic expansion

In this question we will show that the complete three partite complex is a $\frac{1}{100}$-coboundary expander in dimension 1. Let X be $X(0) = A \cup B \cup C$ where A, B, C are three disjoint sets of size $n > 0$ and $X(2) = \{\{a,b,c\} \mid \ a \in A, b \in B, c \in C\}$.

Let $f : X(1) \to \{0,1\}$ be a function so that

$$wt(\delta f) = \frac{|\{\{a,b,c\} \in X(2) \mid \ \delta f(\{a,b,c\}) \neq 0\}|}{n^3} \leqslant \varepsilon.$$

Our goal is to find some $g : X(0) \to \{0,1\}$ so that $\frac{1}{100}\operatorname{dist}(f, \delta g) \leqslant \varepsilon$.

This exercise guides you towards one proof. You may instead think about another proof if you like.

1. Explain in your own words why finding some $g : X(0) \to \{0,1\}$ so that $\frac{1}{100}\operatorname{dist}(f, \delta g) \leqslant \varepsilon$ shows that X is an $\frac{1}{100}$-coboundary expander.

2. Let $G = (L, R, E)$ be a complete bipartite graph with n vertices on every side. Let $f : E \to \{0,1\}$ be a function so that

$$\mathbb{P}_{b_1,b_2 \in B, c_1,c_2 \in C}\left[f(b_1c_1) + f(c_1b_2) + f(b_2c_2) + f(c_2b_1) \neq 0\right] \leqslant \varepsilon,$$

   Where the probability is over sampling $b_1, b_2 \in L$ and $c_1, c_2 \in R$ uniformly at random and independently.

   Show that there exists a function $g : L \cup R \to \{0,1\}$ so that $\operatorname{dist}(f, \delta g) = \mathbb{P}_{b \in L, c \in R}\left[f(bc) \neq g(b) + g(c)\right] \leqslant \varepsilon$.

   Hint: Think about the proof of coboundary expansion for the complete complex.

3. Construct g for $B \cup C$ only as follows.

   (a) Show that

   $$\mathbb{P}_{b_1,b_2 \in B, c_1,c_2 \in C}\left[f(b_1c_1) + f(c_1b_2) + f(b_2c_2) + f(c_2b_1) \neq 0\right] \leqslant 4\varepsilon.$$

   Where the probability is over sampling $b_1, b_2 \in B$ and $c_1, c_2 \in C$ uniformly at random and independently. Conclude that exists $g : B \cup C \to \{0,1\}$ so that $\operatorname{dist}_{BC}(f, \delta g) \leqslant 4\varepsilon$, where $\operatorname{dist}_{BC}(f, \delta g) = \mathbb{P}_{b \in B, c \in C}\left[f(bc) \neq g(b) + g(c)\right]$.

4. Let us extend g to the vertices of A according to the majority vote. That is, for every $v \in A$ we set $g(v) = \operatorname{maj}\{f(uv) + g(u) \mid \ u \in B \cup C\}$. Show that $\operatorname{dist}(f, \delta g) \leqslant 100\varepsilon$:

   (a) Use the item $2b$ to bound the distance between f and δg for edges between B and C (this is supposed to be immediate).

   (b) Let $v \in A$. Show that $f(vu) \neq g(v) + g(u)$ if and only if u doesn't agree with the majority vote for v.

(c) Let $MIN_v \subseteq B \cup C$ be the sets of the vertices that don't agree with the majority vote on v. Show that $\frac{|Min_v|}{8n} \leqslant \frac{|E(MIN_v, B\cup C \setminus MIN_v)|}{n^2}$.

(d) Show that if $uw \in E(MIN_v, B \cup C \setminus MIN_v)$ then either $\delta f(uvw) \neq 0$, or $f(uw) \neq g(u) + g(w)$.

(e) Use the two items above to bound the distance of f and δg over edges that contain a vertex in A. Get the desired bound on $\mathrm{dist}(f, \delta g)$.

3 Agreement and robust testability

Definition 3.1 (agreement testability)**.** Let $\kappa > 0$. Let $C_i \subset \{f : [n_i] \to \{0,1\}\}$ for $i = 1, 2$. We say that $C_1 \otimes C_2$ is κ-agreement testable if for every $w_1 \in C_1 \otimes \{0,1\}^{n_2}$, $w_2 \in \{0,1\}^{n_1} \otimes C_2$, there exists $w \in C_1 \otimes C_2$ such that

$$\kappa \cdot \left(\mathbb{P}_i[w_1(i,\cdot) \neq w(i,\cdot)] + \mathbb{P}_j[w_2(\cdot,j) \neq w(\cdot,j)]\right) \leqslant \mathbb{P}_{i\in[n_1], j\in[n_2]}[w_1(i,j) \neq w_2(i,j)].$$

Definition 3.2 (Robust testability of tensor codes)**.** Fix $C_i \subseteq \{0,1\}^{n_i}$ linear error correcting codes, for $i = 1, 2$. For $f : [n_1] \times [n_2] \to \{0,1\}$, let

$$\mathrm{dist}_{col}(f) = \mathrm{dist}(f, C_1 \otimes \{0,1\}^{n_2}), \qquad \mathrm{dist}_{row}(f) = \mathrm{dist}(f, \{0,1\}^{n_1} \otimes C_2).$$

and

$$d(f) = (\mathrm{dist}_{col}(f) + \mathrm{dist}_{row}(f))/2.$$

The robust testability of $C_1 \otimes C_2$ is defined to be

$$\rho = \min_{f \notin C_1 \otimes C_2} \frac{d(f)}{\mathrm{dist}(f, C_1 \otimes C_2)},$$

and we say that $C_1 \otimes C_2$ is ρ-robustly testable.

1. Prove that if $C_1 \otimes C_2$ is κ-agreement testable, then $C_1 \otimes C_2$ is ρ-robustly testable for $\rho = \frac{\kappa}{(\kappa+1)}$.
2. Bonus: Show that the converse is also true. Namely,if $C_1 \otimes C_2$ is τ-robustly testable then $C_1 \otimes C_2$ is κ-agreement testable, for $\kappa = \frac{2\tau\delta_1\delta_2}{\delta_2 + \delta_1(1+2\tau)}$ (where δ_i is the relative distance of C_i).

References

[OP22] Ryan O'Donnell and Kevin Pratt. "High-Dimensional Expanders from Chevalley Groups." In: *37th Computational Complexity Conference, CCC 2022, July 20-23, 2022, Philadelphia, PA, USA*. Ed. by Shachar Lovett. Vol. 234. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 18:1–18:26. DOI: 10.4230/LIPIcs.CCC.2022.18. URL: https://doi.org/10.4230/LIPIcs.CCC.2022.18.